

Cooperative Game Theory:

Characteristic Functions, Allocations, Marginal Contribution

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1 Introduction

Game theory is divided into two branches, called the *non-cooperative* and *co-operative* branches. The payoff matrices we have been looking at so far belong to the non-cooperative branch. Now, we are going to look at the cooperative branch.

The two branches of game theory differ in how they formalize interdependence among the players. In the non-cooperative theory, a game is a detailed model of all the moves available to the players. By contrast, the cooperative theory abstracts away from this level of detail, and describes only the outcomes that result when the players come together in different combinations.

Though standard, the terms *non-cooperative* and *cooperative* game theory are perhaps unfortunate. They might suggest that there is no place for cooperation in the former and no place for conflict, competition etc. in the latter. In fact, neither is the case. One part of the non-cooperative theory (the theory of repeated games) studies the possibility of cooperation in ongoing relationships. And the cooperative theory embodies not just cooperation among players, but also competition in a particularly strong, unfettered form.

The non-cooperative theory might be better termed *procedural* game theory, the cooperative theory *combinatorial* game theory. This would indicate the real distinction between the two branches of the subject, namely that the first specifies various actions that are available to the players while the second describes the outcomes that result when the players come together in different combinations. This is an important analytical distinction which should become clear as we proceed.

The idea behind cooperative game theory has been expressed this way:

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“Cooperative theory starts with a formalization of games that abstracts away altogether from procedures and . . . concentrates, instead, on the possibilities for agreement. . . . There are several reasons that explain why cooperative games came to be treated separately. One is that when one does build negotiation and enforcement procedures explicitly into the model, then the results of a non-cooperative analysis depend very strongly on the precise form of the procedures, on the order of making offers and counter-offers and so on. This may be appropriate in voting situations in which precise rules of parliamentary order prevail, where a good strategist can indeed carry the day. But problems of negotiation are usually more amorphous; it is difficult to pin down just what the procedures are. More fundamentally, there is a feeling that procedures are not really all that relevant; that it is the possibilities for coalition forming, promising and threatening that are decisive, rather than whose turn it is to speak. . . . Detail distracts attention from essentials. Some things are seen better from a distance; the Roman camps around Metzada are indiscernible when one is in them, but easily visible from the top of the mountain.”

(Aumann, R., “Game Theory,” in Eatwell, J., Milgate, M., and P. Newman, *The New Palgrave*, New York, Norton, 1989, pp.8-9.)

2 Definition of a Cooperative Game

A cooperative game consists of two elements: (i) a set of players, and (ii) a characteristic function specifying the value created by different subsets of the players in the game. Formally, let $N = \{1, 2, \dots, n\}$ be the (finite) set of players, and let i , where i runs from 1 through n , index the different members of N . The characteristic function is a function, denoted v , that associates with every subset S of N , a number, denoted $v(S)$. The number $v(S)$ is interpreted as the value created when the members of S come together and interact. In sum, a cooperative game is a pair (N, v) , where N is a finite set and v is a function mapping subsets of N to numbers.

Example 1 *As a simple example of a cooperative game, consider the following set-up. There are three players, so $N = \{1, 2, 3\}$. Think of player 1 as a seller, and players 2 and 3 as two potential buyers. Player 1 has a single unit to sell, at a cost of \$4. Each buyer is interested in buying at most one unit. Player 2 has a willingness-to-pay of \$9 for player 1’s product, while player 3 has a willingness-to-pay of \$11 for player 1’s product. The game is depicted in Figure 1 below.*

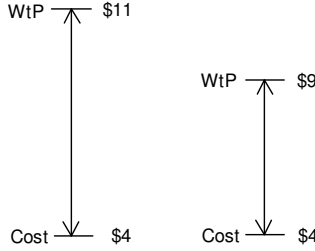


Figure 1

We define the characteristic function v for this game as follows:

$$\begin{aligned}
 v(\{1, 2\}) &= \$9 - \$4 = \$5, \\
 v(\{1, 3\}) &= \$11 - \$4 = \$7, \\
 v(\{2, 3\}) &= \$0, \\
 v(\{1\}) &= v(\{2\}) = v(\{3\}) = \$0, \\
 v(\{1, 2, 3\}) &= \$7.
 \end{aligned}$$

This definition of the function v is pretty intuitive. If players 1 and 2 come together and transact, their total gain is the difference between the buyer's willingness-to-pay and the seller's cost, namely \$5. Likewise, if players 1 and 3 come together, their total gain is the again the difference between willingness-to-pay and cost, which is now \$7. Players 2 and 3 cannot create any value by coming together; each is looking for the seller, not another buyer. Next, no player can create value on his or her own, since no transaction can then take place. Finally, note that $v(\{1, 2, 3\})$ is set equal to \$7, not $\$5 + \$7 = \$12$. The reason is that the player 1 has only one unit to sell and so, even though there are two buyers in the set $\{1, 2, 3\}$, player 1 can transact with only one of them. It is a modeling choice—but the natural one—to suppose that in this situation player 1 transacts with the buyer with the higher willingness-to-pay, namely player 3. That is the reason for setting $v(\{1, 2, 3\})$ equal to \$7 rather than \$5.¹

3 Marginal Contribution

Given a cooperative game (N, v) , the quantity $v(N)$ specifies the overall amount of value created. (In the example above, this quantity was $v(\{1, 2, 3\}) = \$7$.)

¹For more discussion of this last, somewhat subtle point, see “Value-Based Business Strategy,” by Adam Brandenburger and Harborne Stuart, *Journal of Economics & Management Strategy*, Spring 1996, Section 7.1 (“Unrestricted Bargaining”), pp.18-19.

An important question is then: How is this overall value divided up among the various players? (Referring again to the example above, the question becomes: How does the \$7 of value get split among the three players?)

The intuitive answer is that bargaining among the players in the game determines the division of overall value $v(N)$. This bargaining is typically ‘many-on-many.’ Sellers can try to play one buyer off against another, buyers can try to do the same with sellers. Intuitively, a player’s ‘power’ in this bargaining will depend on the extent to which that player needs other players as compared with the extent to which they need him or her. Briefly put, the issue is: Who needs whom more?

The analytical challenge is to formalize this intuitive line of reasoning. This is what the concept of *marginal contribution* does.

To define marginal contribution, a piece of notation is needed: Given the set of players N and a particular player i , let $N \setminus \{i\}$ denote the subset of N consisting of all the players except player i .

Definition 1 *The **marginal contribution** of player i is $v(N) - v(N \setminus \{i\})$, to be denoted by MC_i .*

In words, the marginal contribution of a particular player is the amount by which the overall value created would shrink if the player in question were to leave the game.

Example 1 Contd. *To practice this definition, let us calculate the marginal contributions of the players in the example above:*

$$\begin{aligned} MC_1 &= v(\{1, 2, 3\}) - v(\{2, 3\}) = \$7 - \$0 = \$7, \\ MC_2 &= v(\{1, 2, 3\}) - v(\{1, 3\}) = \$7 - \$7 = \$0, \\ MC_3 &= v(\{1, 2, 3\}) - v(\{1, 2\}) = \$7 - \$5 = \$2. \end{aligned}$$

We will return to this example once more below, to use these marginal-contribution numbers to deduce something about the division of the overall value created in this particular game. Before that, we need to state and justify a principle about the division of value in a cooperative game. (Later on, we will examine a stricter principle than this one.)

Definition 2 *Fix a cooperative game (N, v) . An **allocation** is a collection (x_1, x_2, \dots, x_n) of numbers.*

The interpretation is easy: An allocation is simply a division of the overall value created, and the quantity x_i denotes the value received by player i .

Definition 3 *An allocation (x_1, x_2, \dots, x_n) is **individually rational** if $x_i \geq v(\{i\})$ for all i .*

Definition 4 *An allocation (x_1, x_2, \dots, x_n) is **efficient** if $\sum_{i=1}^n x_i = v(N)$.*

These two definitions are quite intuitive. Individual rationality says that a division of the overall value (i.e. an allocation) must give each player as much value as that player receives without interacting with the other players. Efficiency says that all the value that can be created, i.e. the quantity $v(N)$, is in fact created.

Unless otherwise noted, all allocations from now on will be assumed to be individually rational and efficient.

Definition 5 *An (individually rational and efficient) allocation (x_1, x_2, \dots, x_n) satisfies the **Marginal-Contribution Principle** if $x_i \leq MC_i$ for all i .*

The argument behind the Marginal-Contribution Principle is simple, almost tautological sounding. First observe that the total value captured by all the players is $v(N)$. It therefore follows from the definition of MC_i that if some player i were to capture more than MC_i , the total value captured by all the players except i would be less than $v(N \setminus \{i\})$. But $v(N \setminus \{i\})$ is the amount of value that these latter players can create among themselves, without player i . So, they could do better by coming together without player i , creating $v(N \setminus \{i\})$ of value, and dividing this up among themselves. The putative division of value in which player i captured more than MC_i would not hold up.

The Marginal-Contribution Principle, while indeed almost obvious at some level, turns out to offer a single, far-reaching method of analyzing the division of value in bargaining situations. The power of the principle should become apparent as we explore some applications.

Example 1 Contd. *As a first application, return one more time to Example 1 above. The overall value created was \$7. Let us now use the marginal-contribution calculations above to deduce something about the division of this value among players 1, 2, and 3. Since player 2 has zero marginal contribution, the Marginal-Contribution Principle implies that player 2 won't capture any value. Player 3 has a marginal contribution of \$2, and so can capture no more*

than this. This implies that player 1 will capture a minimum of $\$7 - \$2 = \$5$. The remaining $\$2$ of value will be split somehow between players 1 and 3. The Marginal-Contribution Principle does not specify how this ‘residual’ bargaining between the two players will go.

This answer is very intuitive. In this game, the buyers (players 2 and 3) are in competition for the single unit that the seller (player 1) has on offer. The competition is unequal, however, in that player 3 has a higher willingness-to-pay than player 2, and is therefore sure to win. The operative question is: On what terms will player 3 win? The answer is that player 2 will be willing to pay up to \$9 for player 1’s product, so player 3 will have to pay at least that to secure the product. The effect of competition, then, is to ensure that player 1 receives a price of at least \$9 (hence captures at least $\$9 - \$4 = \$5$ of value); player 3 pays a price of at least \$9 (hence captures at most $\$11 - \$9 = \$2$ of value); and player 2 captures no value. Will the price at which players 1 and 3 transact be exactly \$9? Not necessarily. At this point, the game is effectively a bilateral negotiation between players 1 and 3, in which player 1 won’t accept less than \$9 and player 3 won’t pay more than \$11. The Marginal-Contribution Principle doesn’t specify how this residual \$2 of value will be divided. It allows that player 3 might pay as little as \$9, or as much as \$11, or any amount in-between. A reasonable answer is that the residual \$2 will be split evenly, with player 1 capturing a total of $\$5 + \$1 = \$6$ of value, and player 3 capturing \$1 of value (but other answers within the permissible range are equally acceptable).

We see that the cooperative game-theoretic analysis captures in an exact fashion the effect of competition among the players in a bargaining situation. It makes precise the idea that the division of value should somehow reflect who needs whom more. But the analysis is (sensibly) agnostic on where the bargaining over residual value—what remains after competition has been accounted for—will end up. After all, where this leads would seem to depend on ‘intangibles’ such as how skilled different players are at persuasion, bluffing, holding out, and so on. These are factors external to the game as described in cooperative theory. Thus, an indeterminacy in the theory at this point is a virtue, not a vice.

Cooperative Game Theory: The Core

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1 The Core

Given a cooperative game (N, v) , recall the following definitions from the note “Cooperative Game Theory: Characteristic Functions, Allocations, Marginal Contribution”:

- (i) an *allocation* is a collection (x_1, x_2, \dots, x_n) of numbers;
- (ii) an allocation (x_1, x_2, \dots, x_n) is *individually rational* if $x_i \geq v(\{i\})$ for all i ;
- (iii) an allocation (x_1, x_2, \dots, x_n) is *efficient* if $\sum_{i=1}^n x_i = v(N)$;
- (iv) an (individually rational and efficient) allocation (x_1, x_2, \dots, x_n) satisfies the *Marginal-Contribution Principle* if $x_i \leq \text{MC}_i$ for all i .

Some additional notation will be useful. For any subset S of the set of players N , let $x(S) = \sum_{i \in S} x_i$. In words, the term $x(S)$ denotes the sum of the values received by each of the players i in the subset S .

Definition 1 *An allocation (x_1, x_2, \dots, x_n) is said to lie in the **core** of the game if it is efficient and is such that for every subset S of N we have $x(S) \geq v(S)$.*

Two observations are in order. First, an allocation that lies in the core is individually rational. To see this, let $S = \{i\}$ for some $i = 1, 2, \dots, n$. Note that $x(\{i\}) = x_i$. (Both are the value received by player i .) Thus, the core condition that $x(\{i\}) \geq v(\{i\})$ is precisely the individual rationality condition.

*With the assistance of Amanda Friedenberg and Konrad Grabiszewski. These notes come from “Conceptual Aspects of Added Value,” by Adam Brandenburger and Harborne Stuart, teaching material, 01/04/00. Do not copy or circulate these notes without the permission of the author. core-01-04-07

Second, note that if an allocation lies in the core then certainly it satisfies the Marginal-Contribution Principle. To see this, consider a particular player i and let $S = N \setminus \{i\}$. The core condition says that $x(N \setminus \{i\}) \geq v(N \setminus \{i\})$. The efficiency condition says that $x(N) = v(N)$. But $x_i = x(N) - x(N \setminus \{i\})$ by definition. Putting all this together gives $x_i \leq v(N) - v(N \setminus \{i\})$, which is exactly the condition of the Marginal-Contribution Principle.

In fact, the core can be thought of as a generalization of the Marginal-Contribution Principle. To demonstrate this, we first need to define the marginal contribution of a group of players. (So far, we have considered only the marginal contribution of an individual player.)

Definition 2 *The **marginal contribution** of a subset S of players is $v(N) - v(N \setminus S)$, to be denoted by MC_S .*

(Under this new notation, the marginal contribution of the subset of players consisting of player i alone should be denoted by $\text{MC}_{\{i\}}$. But no confusion will result if we continue to write MC_i for $\text{MC}_{\{i\}}$.)

Theorem 1 *An efficient allocation (x_1, x_2, \dots, x_n) lies in the core if and only if for every subset S of N we have $x(S) \leq \text{MC}_S$.*

Proof. Suppose that the allocation (x_1, x_2, \dots, x_n) is efficient and lies in the core. Then $x(N) = v(N)$ by efficiency. Now consider the subset $N \setminus S$, and use the core condition $x(N \setminus S) \geq v(N \setminus S)$. Since $x(N) = x(N \setminus S) + x(S)$, we can rearrange terms to get $x(S) \leq v(N) - v(N \setminus S) = \text{MC}_S$, as required.

Conversely, suppose that the allocation (x_1, x_2, \dots, x_n) is efficient and satisfies $x(S) \leq \text{MC}_S$ for every subset S of N . Then $x(N) = v(N)$ by efficiency. Now consider the subset $N \setminus S$, and use the condition $x(N \setminus S) \leq \text{MC}_{N \setminus S} = v(N) - v(S)$. Since $x(N) = x(N \setminus S) + x(S)$, we can rearrange terms to get $x(S) \geq v(S)$, as required. ■

Theorem 1 makes clear that the motivations for the Marginal-Contribution Principle and the core are similar. Indeed, the core is another expression of the idea that ‘no good deal goes undone.’ If any group of players, say S , anticipated capturing less value in total than the group could create on its own, i.e. if $x(S) < v(S)$, then the players in this group would do better to create and divide the value $v(S)$ by themselves. This is the ‘good deal’ that can’t go ‘undone’ according to the core, and is why the core imposes the condition that $x(S) \geq v(S)$.

2 Examples

Example 1 Consider a cooperative game with two sellers and two buyers. Each seller is offering to sell one unit of a product. The first seller can make its product available at a cost of \$2. The second seller can make its product available at a cost of \$4. The first buyer has a willingness-to-pay for either product of \$8, and is interested in acquiring only one unit. The second buyer has a willingness-to-pay for either product of \$6, and also is interested in acquiring only one unit.

- (i) What divisions of value are possible in the core?
- (ii) What divisions of value satisfy the Marginal-Contribution Principle?
- (iii) Consider a division of value that satisfies the Marginal-Contribution Principle, but that is not in the core. Provide an argument as to why it is a reasonable outcome, then provide an argument as to why it is not reasonable.

Example 2 Consider a cooperative game with two sellers and three buyers. Each seller has two units to sell at a cost of \$0 per unit. Each buyer is interested in buying one unit at a willingness-to-pay of \$1 for either seller's product.¹

- (i) What divisions of value are possible in the core?
- (ii) What divisions of value satisfy the Marginal-Contribution Principle?
- (iii) What divisions of value do you consider plausible in this game?

Example 3 Consider a cooperative game with two suppliers (labelled s_1 and s_2), two firms (labelled f_1 and f_2), and two buyers (labelled b_1 and b_2). For value to be created, a supplier, firm, and buyer must come together and transact, as follows. Each of the combinations

$$\begin{aligned} &\{s_1, f_1, b_1\}, \\ &\{s_2, f_2, b_1\}, \\ &\{s_2, f_1, b_2\}, \\ &\{s_1, f_2, b_2\}, \end{aligned}$$

creates \$1 of value.

What divisions of value satisfy the Marginal-Contribution Principle?

¹Adapted from Postlewaite, A., and R. Rosenthal, "Disadvantageous Syndicates," *Journal of Economic Theory*, 9, 1974, 324-326.