Enjoying Math: Learning Problem Solving with KenKen® Puzzles

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Dedication

To Mrs. Priya Kulkarni (my wife)

Dr. Shyamkant Kulkarni and Mrs. Rekha Kulkarni (my parents)

without whose support, this book would not have been possible.

To Rahul and Amrita Kulkarni (my children)

whose love of mathematics got me involved in teaching mathematics

ACKNOWLEGEMENTS

I would like to express my thanks to Prof. Harold Reiter for encouraging me to publish this book. This book builds on previous work in creative problem solving, development of puzzles and discovery of different KenKen® solving techniques by many different people. In particular, I was motivated to work on this topic by Prof. Reiter's essays. I have also benefitted from a discussion with Prof. Alan Schoenfeld on methods of teaching creative problem solving. Editorial suggestions from WordSharp Editing, Dr. Patrick Min and Prof. Harold Reiter helped improve this document. I would like to thank Nikki Sato of NexToy, LLC for providing sample KenKen® puzzles for including in this book.

CONTENTS

Preface

Enjoying Math: Learning Problem Solving with KenKen® Puzzles Is a problem solving book for teachers of mathematics published both as an ebook and as a paperback. It is written primarily for Grade 4-7 teachers. Student's motivation for learning mathematical problem solving can be improved dramatically by recreational mathematics. Over the last ten years, I have successfully used puzzles to teach creative problem solving to students. I have found that KenKen® puzzle is exceptionally good in engaging students in mathematical activities that they truly enjoy. The lesson plans I have used have been very effective in improving creative problem solving skills of my students. Some of these students have been top scorers in national mathematics competitions in the United States. At the same time, my students have found math classes to be a very enjoyable experience. Through this book, I would like to share a wide variety of my lesson plans with a wider audience. Additional material that supplements this book would be made available at the book website, http://www.matholympiad.info/Pages/EnjoyingMathBook.aspx. I would like to invite other teachers to provide feedback to me at EnjoyingMath@gmail.com, both about their successes in teaching creative problem solving and about ways to improve this book. Collectively, we can improve the experience of learning mathematics for our students.

Deepak Kulkarni

The Joy of Creative Problem Solving

From time immortal, people have enjoyed activities such as games, magic shows, contests, and puzzles. Therefore, it is not surprising to find students enjoying similar activities based on mathematics. There are a wide variety of math-based games and math game software. Good examples of math games include Krypto and 24. Like games, contests have an appeal to many kids who enjoy the process of doing things to win something. Therefore, math contests can be an activity that kids enjoy and that can encourage kids to work on math problems. In the process of taking part in contests, some kids begin to love math. Math competitions in which students can participate in the United States include NOETIC Learning Math Contest, MOEMS, Math Bee by North South Foundation, Math Kangaroo, World Math Day, Ole Miss Math Challenge, Online Math League, MATHCOUNTS and AMC. Math magic tricks include tricks about guessing numbers and some card tricks based on math.

Yet another entertaining activity is doing math puzzles. This book will examine a variety of math techniques in the context of math puzzles. In particular, we will be studying creative problem solving in the context of a puzzle called KenKen®. We enjoy working on puzzles because we have a natural tendency to be motivated by surprise, contradiction and a gap in knowledge. While a math puzzle can intrigue and engage students and get them going, a challenging, questioning and reflecting atmosphere can make the experience of mathematical problem solving even more enjoyable.

With the right attitude and practice, students can enjoy the process of mathematical thinking. This process involves thinking about mathematical problems, observing beautiful mathematical patterns, coming up with elegant insights, facing difficult problems that one may or may not be able to solve, experiencing the thrill of progressing on such problems and solving them, reflecting on mathematical thinking, and learning from successes and failures. Once students begin to love creative math problem solving, they have an activity they can enjoy wherever they are. Then, the joy of creative thinking is all they need to motivate themselves to get going on any challenging math problem.

The Problem Solving Approach

For some problems, students know the strategy to use as soon as they read the problem. However, for particularly difficult problems, they do not know right away how they can solve them. The progress on such problems often comes from heuristics or 'rules of thumb' that are likely to be useful, but are not guaranteed to solve problems. As a result, the progress on a problem takes the form of multiple explorations or searching different ideas. Work on the problem solving may go through different phases such as trying to understand the problem, working on a specific approach, being stuck and trying to get unstuck, critically examining solutions, or communicating. The work may involve going back and forth between these different phases of work. In this book, we would now be providing a variety of different rules of thumb for solving problems. These heuristics can be described in the form of a condition and an associated action, where conditions describe problem situations and actions describe what should be done in such situations.

Situation: Are you about to start working on a problem? Are you trying to understand a problem?

Try to understand the problem by asking the following:

What is given and what is to be found? Is it possible to draw a picture or a diagram of the context described in the problem? Can you reword the problem? Can you come up with specific examples corresponding to the problem?

Situation: Have you thought out an approach to attack the problem?

If the general approach to solving the problem is obvious to you, create a plan to solve the problem based on this approach and carry out this plan.

If you know a related or similar problem, you can use the knowledge of the solution from the related problem to come up with a plan.

If you can't formulate an approach, you may be feeling stuck and you may want to try to understand the problem better.

Situation: Are you feeling stuck?

Many different approaches can be tried to get unstuck. One approach is to try working a simpler version of the problem, and use the solution to the problem to get insights that are useful in solving the original problem.

When you come up with a pattern or an 'Aha' moment, try studying the observations that triggered it in more detail and try observing how these could be used in progressing with the problem.

Alternatively, you may just try to understand the problem better and use relevant suggestions.

Situation: Are you busy working out details?

Monitor how you are progressing and backtrack if needed.

Do not forget to look for patterns, the unusual and surprises (Aha! insights).

Look for any surprise; understand it and its implication for the problem.

Situation: Are you done solving a problem or a sub-problem or inferring a key conclusion?

Critically examine your hypotheses and solutions.

Done solving the problem? If it works, check each step. Can you see clearly that the step is correct? Can you prove that it is correct?

Learn from reflection: Specialize/generalize heuristics. Learn new heuristics. If the plan does not produce a solution in a short time, then check from time to time: why are you doing what you are doing? Are you progressing? This is self-monitoring. If your plan fails, examine why it did not work. Writing with a rubric or a template can help in recalling and studying what you have done so far. Organize the information. Ask: What can you conclude about the approaches that won't work? What else did you learn? Do you see any patterns?

Situation: Are you about to communicate your conclusions to a teacher or to partners?

The final part of your work on a problem is to communicate your conclusions. What is communicated may differ depending on the situation. Sometimes, you are expected to report only the answer to the problem. Sometimes, you are expected to show your work. Sometimes, you may be doing collaborative problem solving. In such situations, it is important to be a good communicator. Helping others with problems that you have solved can help you develop skills needed to become a good math communicator. The aspects of such communication include explaining your solution to someone else clearly, understanding someone else's solution, and providing feedback on it at various levels of detail. After you create an explanation for your solution, examine carefully if you have justified each step in the work.

Specific Problem Solving Strategies

1. Change the representation

Using a wrong representation may make a problem impossible to solve. Strategies of changing representation include drawing a picture and looking at the problem from a completely different perspective. By drawing a picture, and visualizing the information about the problem using it, you will have clearer understanding of the problem and it will help you to come up with an approach to solve the problem that you might not be able to see otherwise.

2. Make an organized list or a table

Making an organized list allows you to examine data clearly. It can help you in ensuring that you are looking at all of the relevant information. It will also allow you to see patterns in the data easily and to come to correct conclusions. Similarly, making a table allows you to examine data clearly. It can help you in ensuring that you are looking at all of the relevant information. It also will allow you to see patterns in the data easily and to come to correct conclusions.

3. Create a simpler problem

Sometimes we are not able to solve the problem as it is stated, but we are able to solve a simpler problem that is similar in some way. For example, the similar problem may use simpler numbers. Once we solve one or more simpler problems, we may understand the approach that can be used to solve the problems of similar type and may be able to solve the problem that has been given to us.

4. Use logical reasoning

Logical reasoning is useful in mathematics problem in various ways. It can be used to eliminate possible choices. It can also sometimes be used to conclude the answer directly.

5. Guess and check

The ‘guess and check’ strategy can be used on many problems. If the number of possible answers is small, one can use this strategy to come up with the answer very quickly. In some other cases where the number of possible answers is not small, one may still be able to make intelligent guesses and come up with the answer.

6. Work backward

Sometimes, it is easier to start with information at the end of the problem and work backward to the beginning of the problem than the other way around.

The Right Attitude toward Working on Difficult Problems

Often, when one is not able to solve a problem, one feels frustrated. The natural tendency is to be disappointed, as 'ego' feels hurt. At an early stage of the problem solving process, one may be stuck while solving a problem. As you are stuck, you may not know of any action you can take to make progress on the problem. However, you may believe that the teacher is expecting you to do some work. Therefore, you feel unhappy about the situation. Furthermore, when you are stuck and not able to think of ways to progress, you anticipate that you are likely to fail in solving the problem. This adds to the frustration of the situation. This explains why it is common to see students with a negative attitude toward difficult problems.

Attitudes that help students enjoy work and persist in effort include some of the following elements:

1) Acceptance of the process: Acceptance of the process of solving difficult problems in which you work for a long time and you are not always sure if you will be able to solve the problem and that 'being stuck' is a normal state and that such a process includes mixed emotions.

2) The thrill of taking on challenges: When one works on an easy task, not solving it is viewed as something of concern whereas solving it is not a big accomplishment. In contrast, when one works on a challenging problem, not solving it is not a concern, as the problem is inherently difficult for anyone. When one does solve a challenging problem, there is tremendous satisfaction and a sense of accomplishment. Despite this, it is natural to feel frustrated when you are stuck. When this happens, you can start by trying to identify what is difficult about the problem and writing down information about the stuck state. Learn a few approaches (e.g., try a simpler problem) that can always be used when you are stuck and when you don't know what approach you can try next. Initially, keep the goal 'to try to make progress on solving the problem' instead of setting the goal of completely solving what seems like a very difficult problem. Thus, one would set many short-term objectives in the process of solving a difficult problem and one would succeed in many of these even if one does not succeed in the overall goal. In particular, when you use the strategies of working on a simpler version of the problem or working on specialized cases of the problem, realize that you are actually solving some problems in the process and making progress. Making progress involves gathering information, noticing patterns and gaining insights about the problem. This way, you would have a sense of accomplishment if you work on the problem and progress without completely solving the problem.

Sometimes, after initially feeling frustrated, one is able to make progress on the problem and solve the problem.

3) Attitude toward failures: Do not be discouraged by failures. Read this quote from the famous scientist, Edison. An assistant asked, “Why are you wasting your time and money? We have had failure after failure, almost a thousand of them. Why do you continue to pursue this impossible task?” Edison said, “We haven’t had a thousand failures, we’ve just discovered a thousand ways not to invent the electric light.” Failure often offers a bigger opportunity for learning than successes.

   Thirst for learning, furthermore, has a clear objective of trying to learn from successes and failures in problem solving process. To learn the most, you need to reflect on both successes and failures. In addition, you are going to learn the most if you are working on the kind of problem that you are not always capable of solving.

4) Appreciation of beauty in mathematics: Appreciate particularly neat insights and your ‘Aha’ moments as you progress on problem solving. These may be interesting patterns and surprises you encountered in problem solving. Ingredients of beauty in mathematics include surprise at the unexpected, the perception of unsuspected relationships and alternation of perplexity and illumination. Mathematical beauty is found in patterns. Famous mathematician Hardy wrote, “A mathematician, like a painter or a poet, is a maker of patterns. If his patterns are more permanent than theirs are, it is because they are made with ideas. The mathematician’s patterns, like the painter’s or the poet’s, must be beautiful; the idea, like the colors or the words, must fit together in a harmonious way.”

5) Interest in mathematical communication: It helps to write the insights you learn as you work on the problem and those you learn when you reflect on your successes and failures. Communicating about these two to others helps as well. If you learn a mathematical trick or a puzzle in class, you may want to share it with your friends or siblings.

Beliefs

Students often hold beliefs about the nature of mathematics that hinder their ability to solve difficult problems creatively. Examples of such misleading beliefs include the following:

- There aren’t multiple ways to solve a problem.
- Average students cannot expect to understand mathematics.
- Mathematics problems are invariably solved by individuals and not by a group of people.
- Students who excel at mathematics solve any problem in a very short time.
- The mathematics topics studied in school are not useful in the real world.

Learning From Reflection

The more you practice the better you will be. However, practice alone is not enough. Reflection over problem solving experience can help a student learn both about the problem situation and about problem solving process.

Recollect how you progressed toward the solution. Remember the important aspects of the progress. Remember the stages when you were stuck and how you recovered. Also, remember the ‘Aha’ moments if you encountered any.

What can you learn from your experience? What made the problem difficult? What worked? What did not work? What was the lesson learnt? Does it tell you about effectiveness of different approaches to problems of this type? If you articulated particular rules of thumb or strategies, what is the reason these worked? In what circumstances would these work? Are these specific cases of more general strategies?

An important part of reflecting on your problem solving experience is to get a better understanding of strategies and rules of thumb that would be useful in future problem situations and, if possible, to come up with new rules of thumb. This includes getting a better understanding of circumstances under which a heuristic would be applicable as well as specializing or generalizing heuristics.

Influence of Parents and Friends

Friends and parents play a very important role in helping kids develop positive attitudes toward mathematics.

Kids would often be motivated to attend school math clubs because they get to spend time with their friends. If the club offers snacks, that may provide additional motivation. Math clubs do encourage positive attitudes toward math and contribute to a higher level of success in mathematics. If you have a child who has strong interest with mathematics and who does not have friends with similar interests in class, it would be helpful to encourage him/her to participate in the Math Club if your school has one. Other alternatives include enrolling him/her in GATE math classes where he or she gets to interact with kids with similar interests. Summer math camps can serve this purpose as well.

Parents can play an important role in encouraging students to take interest in math by doing and supporting math at home. Two sites outlining useful recommendations for parents are listed below:

Kanter, P and Darby, L. Help Your Child Learn Math. 1999

http://www2.ed.gov/pubs/parents/Math/title.html

Core-Plus Mathematics Project. Questions to Ask Children When Helping a Student. 2012

http://www.wmich.edu/cpmp/parentresource2/questions.html

Use of Grid Diagrams

Consider following puzzle:

I have sixteen numbers: A1, A2, A3, A4, B1, B2, B3, B4, C1, C2, C3, C4, D1, D2, D3, and D4. I have two sets of clues about these numbers.

Clue set 1:

A1, A2, A3 and A4 are the numbers 1 to 4, but not necessarily in that order.
B1, B2, B3 and B4 are the numbers 1 to 4, but not necessarily in that order.
C1, C2, C3 and C4 are the numbers 1 to 4, but not necessarily in that order.
D1, D2, D3 and D4 are the numbers 1 to 4, but not necessarily in that order.
A1, B1, C1 and D1 are the numbers 1 to 4, but not necessarily in that order.
A2, B2, C2 and D2 are the numbers 1 to 4, but not necessarily in that order.
A3, B3, C3 and D3 are the numbers 1 to 4, but not necessarily in that order.
A4, B4, C4 and D4 are the numbers 1 to 4, but not necessarily in that order.

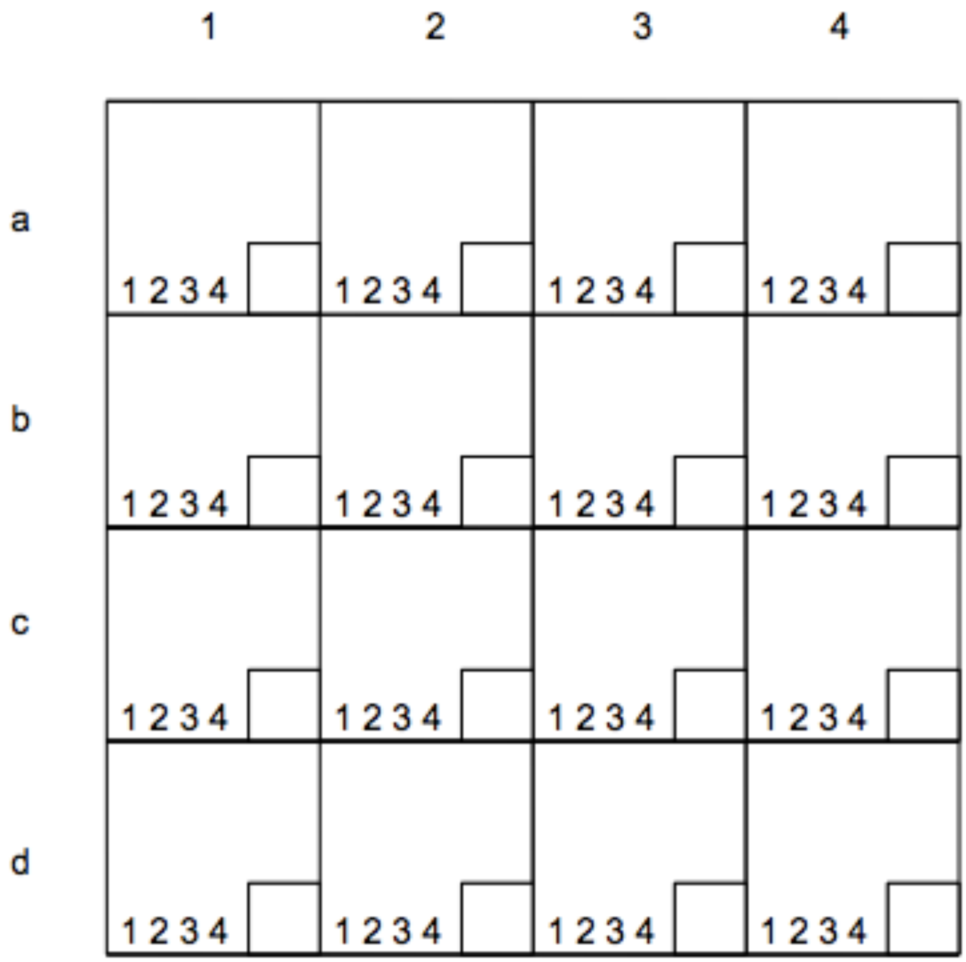
Clue set 2:

Either A1 – B1 = 2 or B1 – A1 = 2
Either C1 divided D1 is 2 or D1 divided by C1 is 2.
Either A2 – B2 = 3 or B2 – A2 = 3
C2 = 2
A3 x A4 x C3 = 16
C3 + C4 + A4 = 7
Either D2 – D3 = 2 or D3 – D2 = 2
D4 = 2

You will find that reasoning about these clues is very hard because there are many different unknowns and many different clues. In some problems, using an appropriate diagram allows us to make inferences that we would otherwise not be able to do.

Now, do the following exercise:

Represent above puzzle information in the grid shown below:



A

An example of a diagrammatic representation of this type puzzles is the KenKen® puzzle.

EXERCISES

In the following exercises, clue set 1 is still applicable, but clue set 2 is replaced by a new set of clues that is described. Represent these puzzles using grid diagrams and try to come up with a solution for each.

1)

Sum of a1, b1 is 5.

Difference between a2, a3 in some order is 2.A4 is 3
Product of b2, c2 is 12.Sum of b3, c3 is 4.
Difference between b4, c4 in some order is 1.
c1 is 2
Product of d1, d2 is 3.
Difference between d3, d4 in some order is 2.
Sum of a1, b1 is 5.
Difference between a2, a3 in some order is 2.
A4 is 3
Product of b2, c2 is 3.
Sum of b3, c3 is 3.
Difference between b4, c4 in some order is 2.
c1 is 3
Product of d1, d2 is 8.
Difference between d3, d4 in some order is 2.

(2)

Sum of a1, b1 is 5.
Difference between a2, a3 in some order is 2.
A4 is 3
Product of b2, c2 is 3.
Sum of b3, c3 is 5.
Difference between b4, c4 in some order is 3.
c1 is 2
Product of d1, d2 is 12.
Difference between d3, d4 in some order is 1.

(3)

Sum of a1, b1 is 5.
Difference between a2, a3 in some order is 2.
A4 is 3
Product of b2, c2 is 3.
Sum of b3, c3 is 7.
Difference between b4, c4 in some order is 1.
c1 is 2
Product of d1, d2 is 6.
Difference between d3, d4 in some order is 3.

(4)

Sum of a1, b1 is 5.
Difference between a2, a3 in some order is 2.
A4 is 3

Product of b2, c2 is 4.
Sum of b3, c3 is 5.
Difference between b4, c4 in some order is 1.
c1 is 3
Product of d1, d2 is 6.
Difference between d3, d4 in some order is 3.

(5)

Sum of a1, b1 is 5.
Difference between a2, a3 in some order is 2.
A4 is 3
Product of b2, c2 is 6.
Sum of b3, c3 is 4.
Difference between b4, c4 in some order is 3.
c1 is 2
Product of d1, d2 is 3.
Difference between d3, d4 in some order is 2.

(6)

Sum of a1, b1 is 5.
Difference between a2, a3 in some order is 2.
A4 is 3
Product of b2, c2 is 6.
Sum of b3, c3 is 5.
Difference between b4, c4 in some order is 1.
c1 is 3
Product of d1, d2 is 2.
Difference between d3, d4 in some order is 1.

SOLUTIONS

(1)

1	2	4	3
4	3	1	2
3	1	2	4
2	4	3	1

(2)

1	2	4	3
4	3	2	1
2	1	3	4
3	4	1	2

(3)

1	4	2	3
4	1	3	2
2	3	4	1
3	2	1	4

(4)

1	2	4	3
4	1	3	2
3	4	2	1
2	3	1	4

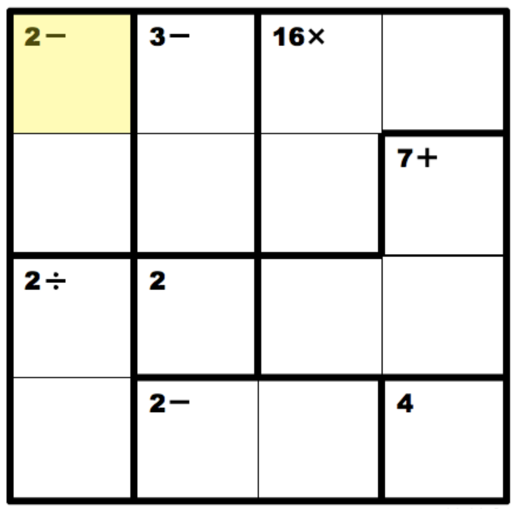
(5)

1	4	2	3
4	2	3	1
2	3	1	4
3	1	4	2

(6)

1	4	2	3
4	3	1	2
3	2	4	1
2	1	3	4

KenKen® Puzzles



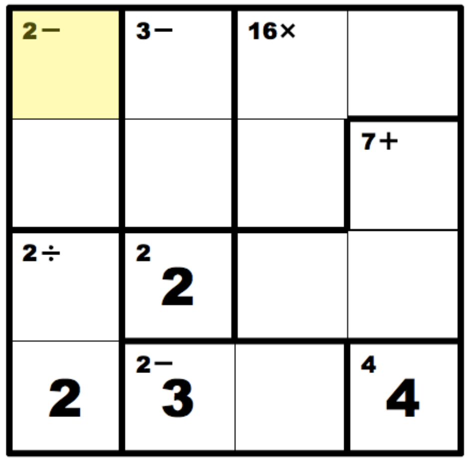
The image above is an example of a KenKen® puzzle. The rules of this puzzle are as follows:

- The 4×4 puzzle uses the numbers from 1 to 4, the 5×5 puzzle uses the numbers from 1 to 5, and so on.
- Each digit can appear only once in each row and in each column.
- The blocks with thick borders are called cages. Each cage shows a result and a mathematical operation. A mathematical operation can be addition, subtraction, multiplication or division. The operation applied to the numbers in the cage should produce the target number. Order is not fixed in the case of subtraction and division.
- A cage with just one square should be filled with the target number associated with the cage.
- A number can be repeated in a cage as long as it is not in the same row or column.
- Each KenKen puzzle has only one solution. Thus, there is only one possible way to put numbers in the squares in the puzzle.

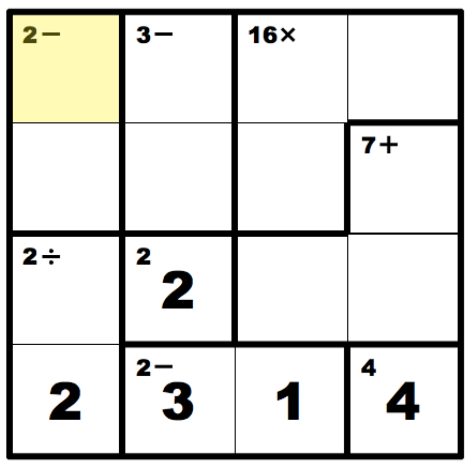
In the puzzle from above, we may first fill the one square cages.

Next, we may conclude that the 2- cage in the bottom row must have 1 and 3. Therefore, the bottom left cage must have 2.

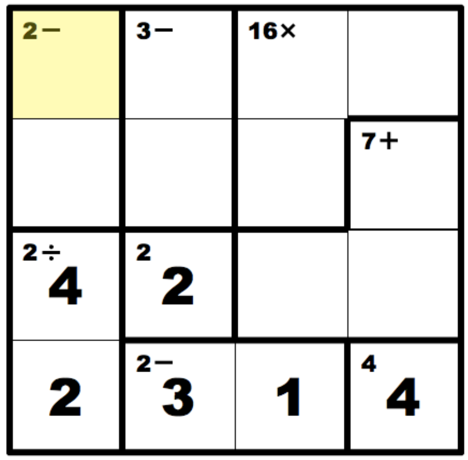
As the 3- cage in the second column must have 1 and 4 and the second column must have the numbers 1, 2, 3 and 4, the second square in the bottom row must have 3.



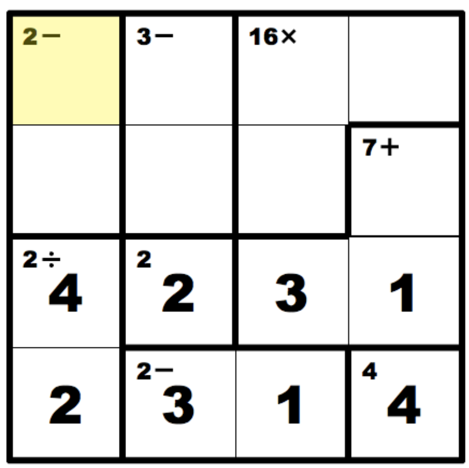
As the bottom row must have all numbers 1, 2, 3 and 4, the third square in the bottom row must have 1.



The third square in the first column can have either 1 or 4 as the target of the ratio of 2 is consistent with either case. However, if the number 1 is in the third square of the first column, the remaining two numbers won't be consistent with the 2- target for the top two squares in the first column. Hence, the third square in the first column cannot be 1 and it must be 4.

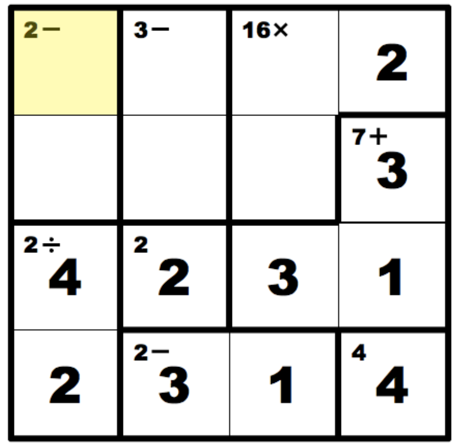


The remaining numbers in the third row must be 1 and 3. The third number in that row can't be 1 as we already have a 1 in the third column. Therefore, the third square in the third row must be 3. Now, the only remaining number in the third row is 1 and that must go in the fourth square in the third row.

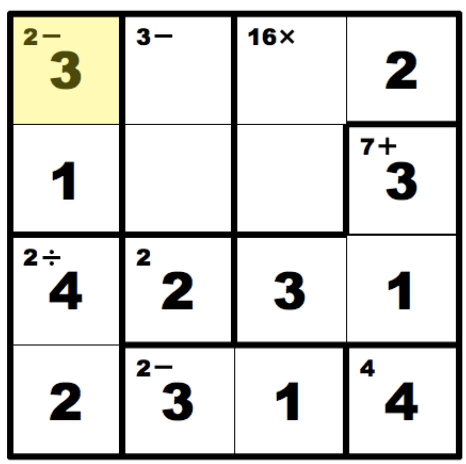


The remaining number in the 7+ cage must be 7 - 1 - 3 = 3 given that it is the cage with a target of 7+.

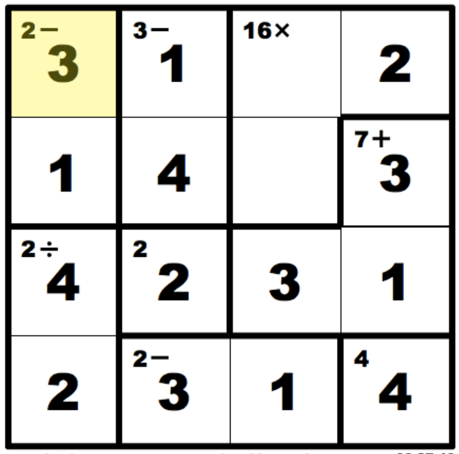
The only unassigned number in the last column is 2 and it must go in the remaining square.



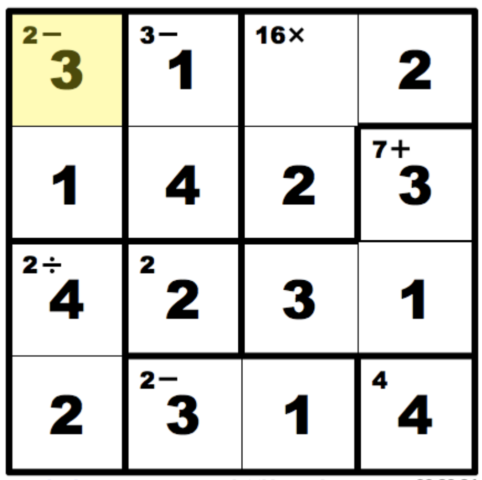
Second square in the first column must have 1 as the column already has 2 and 4 and the row already has 3. Then, the first square in that column must have 3 to be consistent with the cage target of 2-.



Second square in the second column must have 4 as the second column already has 2 and 3 and the second row already has 1. Next, the first square in the second column must have 1 to be consistent with the target of 3-.



Next, third square in the second row must have 2.



The remaining square in the puzzle must be 4 as both the corresponding rows and columns have 1, 2 and 3. BINGO! We have now solved the puzzle. We can pause and appreciate the joy of gradually progressing on a puzzle and finally solving it.

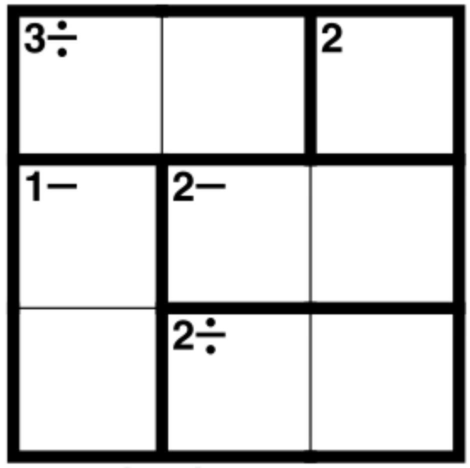
When we are reasoning about a part of a KenKen® puzzle, the problem can be viewed as a problem about a list of numbers with certain constraints. For example, in the above puzzle, we can create a problem corresponding to the bottom row of the puzzle that may read as follows:

I have four numbers, 1, 2, 3 and 4, but not necessarily in the same order. The last number is 4. The difference between the second and the third number is 2. What is the value of the first number?

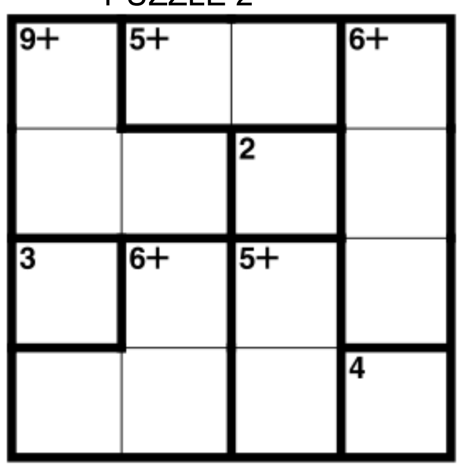
For the puzzles below, solve the puzzle and create a detailed explanation similar to that given above.

EXERCISES

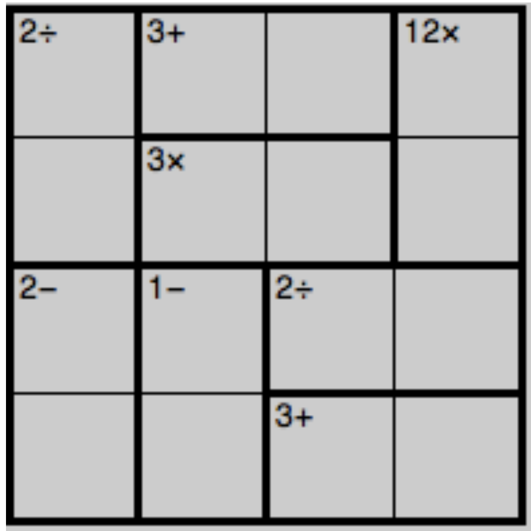
PUZZLE 1



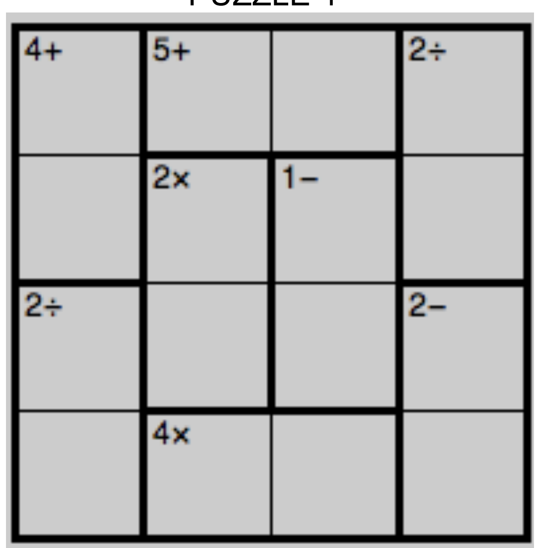
PUZZLE 2



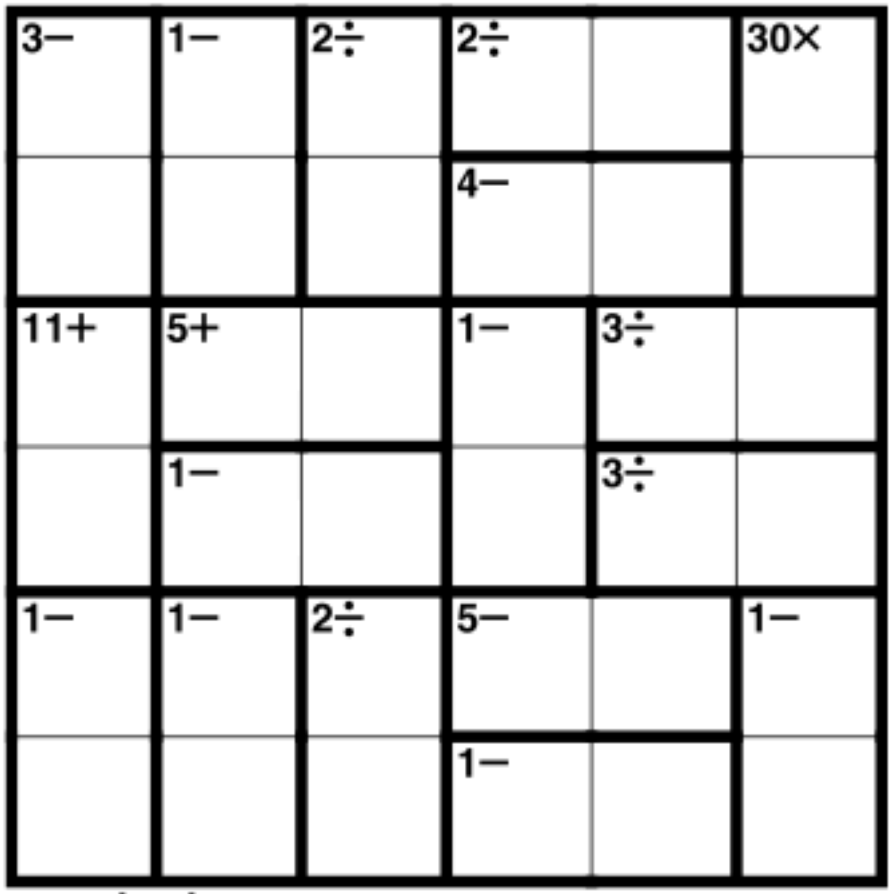
PUZZLE 3



PUZZLE 4



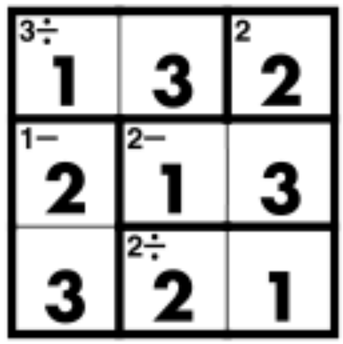
PUZZLE 5



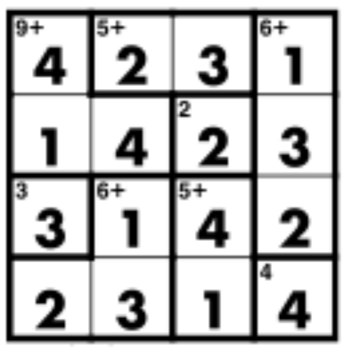
6) Reflect on the things you learnt in this exploration. Write down some things you learnt.

SOLUTIONS TO PUZZLES

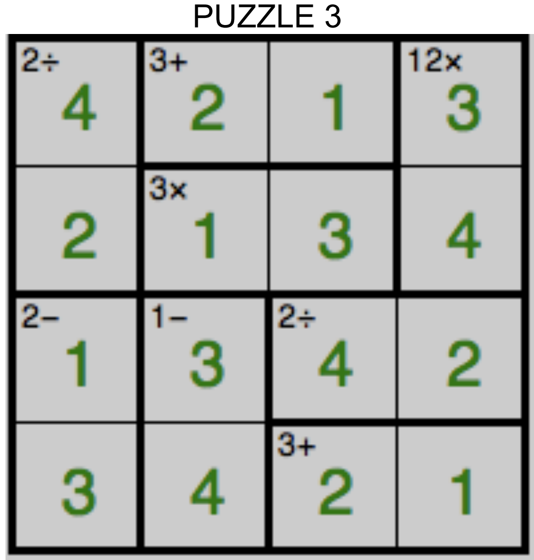
PUZZLE 1



PUZZLE 2



PUZZLE 3



PUZZLE 4

4+ 1	5+ 3	2	2÷ 4
3	2× 1	1− 4	2
2÷ 4	2	3	2− 1
2	4× 4	1	3

PUZZLE 5

3− 1	1− 3	2÷ 6	2÷ 4	2	30× 5
4	2	3	4− 1	5	6
11+ 5	5+ 1	4	1− 3	3÷ 6	2
6	1− 4	5	2	3÷ 3	1
1− 3	1− 5	2÷ 2	5− 6	1	1− 4
2	6	1	1− 5	4	3

Guess and Check Strategy

A strategy that many students attempt to use on puzzles is that of trial and error or guess and check.

Guess and check often involves following steps.

- Figure out all the unknown quantities that are potential candidate values one can guess.
- Figure out the range of possible values of these quantities.
- Of these, try to figure which values are easier to guess. Often, these quantities are the ones that can take fewer possible values.
- Use constraints given in the problem to figure out values of other quantities.
- See if there are any contradictions. If there is, you need to guess another value and repeat the process.
- In guessing new value, see if you can information about what happened with your previous guesses.

Suppose, we have been asked to find out a pair of two different numbers between 1 and 4 that add up to 7. Here, steps we would follow would be as follows

- First, we figure out that two unknown quantities are the first number and the second number.
- We have told that the values these take are from 1 to 4.
- We choose to guess the first number and guess it to be 1.
- We have told that the two numbers add up to 7. So, the second number must be 6.
- This contradicts the information that the second number must be from 1 to 4. So, we need to guess again. We figure that our second guess must be higher.
- After a couple of attempts, we guess the first number to be 3 and find out that the second number can be 4.
- So, the two numbers are 3 and 4.

Now, consider the problem of finding all pairs of two numbers between 1 and 6 that add up to 7. ‘Guess and check’ can only tell you whether specific guesses meet the conditions specified in a problem, but it does not tell us whether we have a complete of answers to a question. If we attempt some KenKen® puzzles, we will find that 'guess and check' strategy can lead us to choose numbers for cells that meet the conditions of a particular cage, but are not the only solutions satisfying that cage. As a result, sometimes we may find later on that solution we have is not the correct solution to a problem. It is very important to remember the following:

If you come with a guess that is consistent with one clue, that does not mean that there aren't other numbers consistent with the clue or that your guess as a part of your overall solution is consistent with all clues in a given KenKen®.

EXERCISES

1) Use guess and check to come up with two numbers from 1 to 5 that add up to 9.
2) Use guess and check to come up with two numbers from 1 to 5 whose product is 20
3) Use guess and check to come up with two numbers from 1 to 4 whose product is 8
4) Use guess and check to come up with two numbers from 1 to 4 whose product is 12
5) Use guess and check to come up with two numbers from 1 to 4 whose product is 12
6) Use guess and check to come up with two numbers from 1 to 4 whose product is 2
7) Use guess and check to come up with two numbers from 1 to 4 whose sum is 3
8) Use guess and check to come up with two numbers from 1 to 4 whose product is 3
9) Use guess and check to come up with two different numbers from 1 to 4 whose product is 4

SOLUTIONS

1) 4, 5
2) 4, 5
3) 2, 4
4) 3, 4
5) 3, 4
6) 1, 2
7) 1, 2
8) 1, 3
9) 1, 4

Draw a smiley in the circle below to celebrate your progress on this exploration.

Exploration of Detecting Mistakes in Logical Reasoning

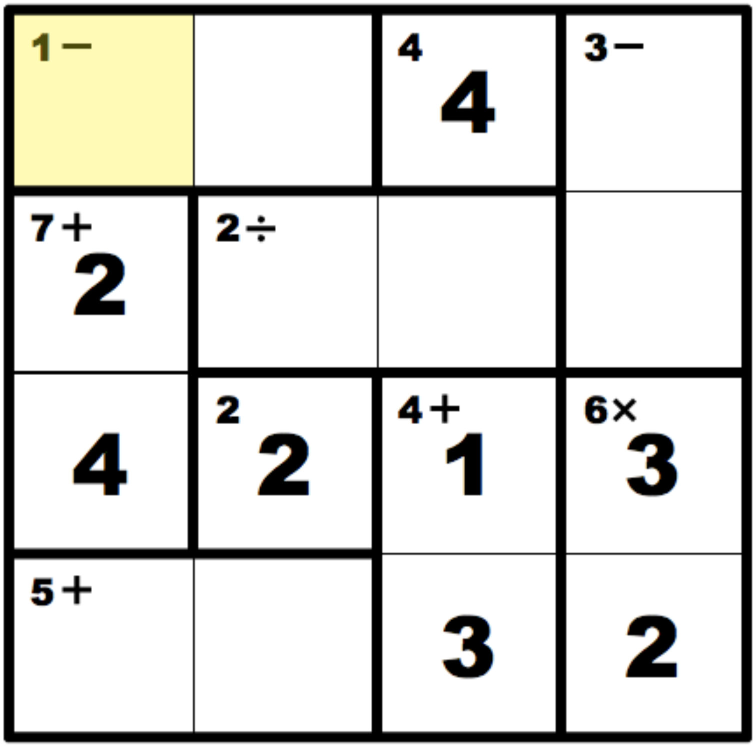
There are two kinds of mistakes in reasoning about KenKen® puzzles.

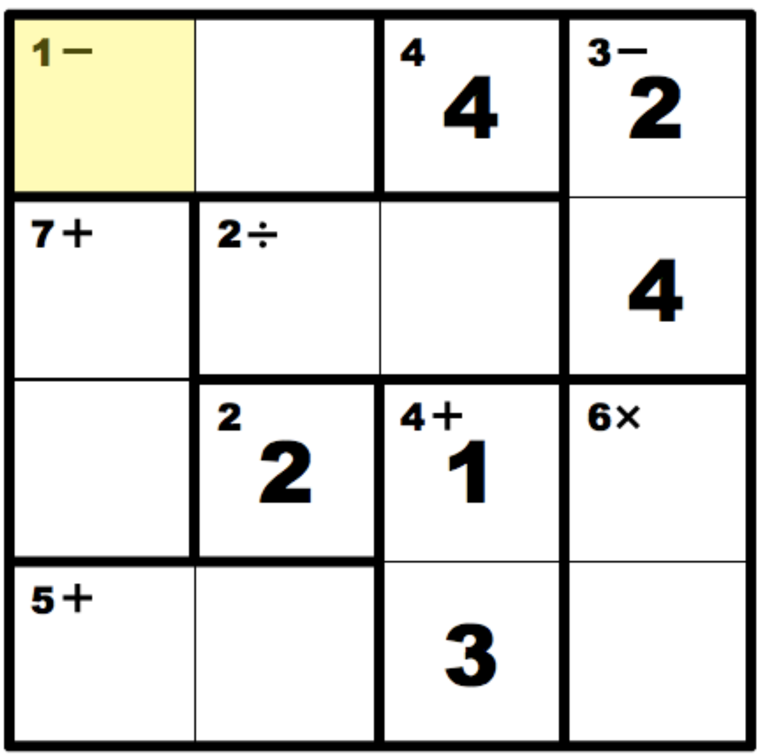
One may make an accidental mistake ('silly mistake') in carrying out a mathematical operation such as addition, multiplication, subtraction or division. For example, we have a cage constraint "3+", if we misread this as "3x" and enter 3 and 1 as a solution. The primary method of detecting mistakes of this kind is to redo the calculations carefully and check if the numbers you entered in KenKen® grid indeed satisfy the cage constraints.

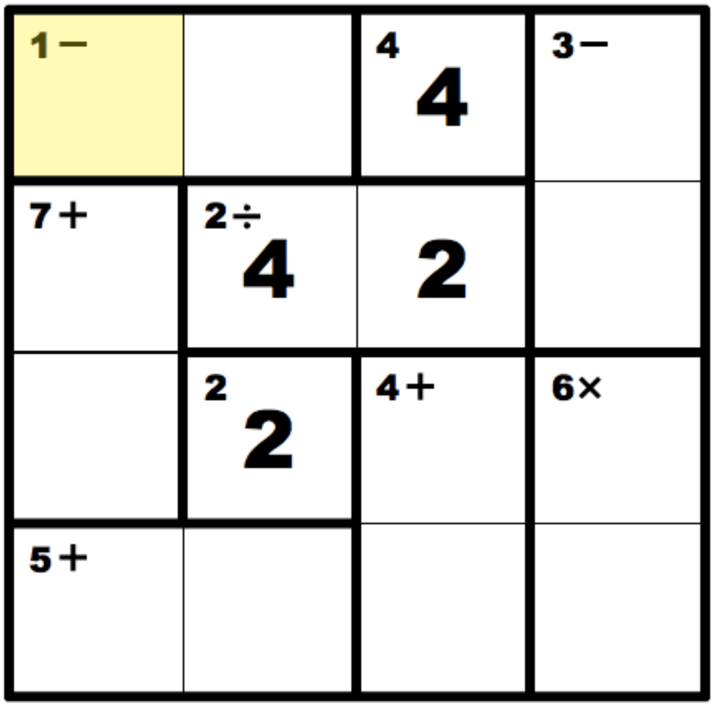
Other type of mistake is in assuming that a particular solution that satisfies a particular cage constraint is the only solution that does so. For example, we have "2/" as a clue and if we were to infer that the solution consists of 1 and 2 as the two numbers, we are jumping to a conclusion. The primary method of detecting mistakes of this kind is to ask ourselves if there is another solution that is consistent with all constraints for a given cage.

EXERCISES

Detect mistakes in the following KenKen® puzzle entries:


1− 4 4 3−
7+ 2 2÷
4 2 2 4+ 1 6× 3
5+ 3 2


1− 4 4 3− 2
7+ 2÷ 4
2 2 4+ 1 6×
5+ 3



SOLUTIONS

1) Entry of 2 and 4 in 7+ cage is not correct.
2) Entry of 2 and 4 in 3- cage is not correct.
3) 4 and 2 are not the only entries that can be entered in ‘2/’ cage at this point.

Exploration of Sets and Venn Diagrams

A *set* is a collection of things. For example, the items you wear are a set: these would include skirt, socks, hat, shirt, jeans, and so on. You write sets with curly brackets like this: {skirt, shoes, jeans, watches, shirts, ...}

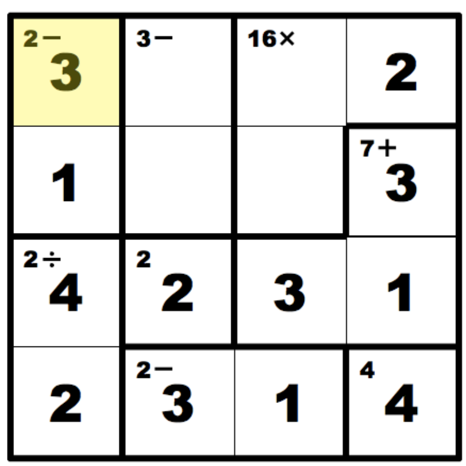
The *union* of two sets is the set of elements that are in either set. For example: let A = {1, 2, 3} and let B = {3, 4, 5}. The union of A and B is written as A U B = {1, 2, 3, 4, 5}. There is no need to list the 3 twice.

The *intersection* of two sets is the set of elements that are in both sets. For example: let A = {1, 2, 3} and B = {3, 4, 5}. The intersection of A and B is written as $A \cap B = \{3\}$. Sometimes there will be no intersection at all. In that case, we say the answer is the *empty set* or the *null set*. For example, given set A = all prime numbers greater than 5 and set B = all even prime numbers, then the intersection of A and B = {}.

The *difference* between A and B are elements that are in A but not in B.

A = {1, 2, 3} B = {3, 4} Then, A – B = {1, 2}

Now, consider the following KenKen® puzzle.



Consider the following sets

S1 = The set of numbers that are not assigned in the second row.

S2 = The set of numbers that are not assigned in the second column.

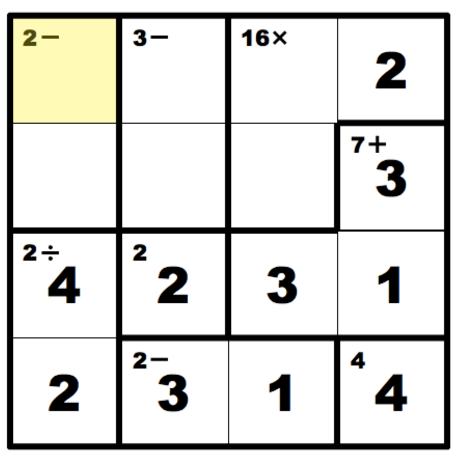
S3 = The set of numbers that are consistent with 3- cage constraints.

S4 = The set of numbers that are assigned in the second row.

S5 = The set of numbers that are assigned in the second column.

EXERCISES

1. Identify S1, S2, S3, S4 and S5.
2. Find the intersection of S1 and S2.
3. Find the intersection of S1, S2 and S3.
4. What can you conclude about the second square in the second row from your answer to (3)?
5. Represent S1, S2 and S3 in a picture.
6. Find the union of S4 and S5.
7. There are three sets F, R and C. The union of the three sets has 60 members. F has 32 members. R has 32 members. C has 22 members. The intersection of F and C has 10 members. There are 10 members that are exclusively in C. There are 16 members exclusively in R. There are 6 members in the intersection of F, R and C. How many members are exclusively in F?
8. Examine the KenKen® puzzle below.



Identify the following sets.

S1: The set of unassigned values in column 1

S2: The set of unassigned values in row 2

Find the intersection of S1 and S2. Use this to determine the value in the second square in the first column.

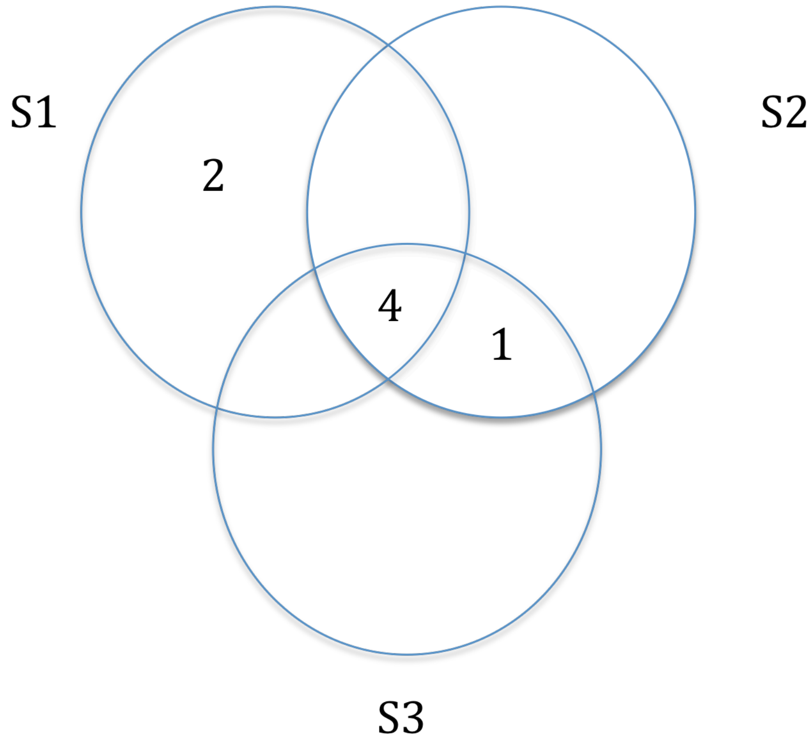
9. The tenth row of a 10x10 KenKen® puzzle has the following numbers unassigned: {1, 2, 3, 4, 5}. The tenth column has the following numbers unassigned: {3, 4, 8, 9}. What are the possible numbers that can go in the square at the intersection of the tenth row and the tenth column?
10. S1 has 10 members. S2 has 8 members. Their union has 16 members. How many members does their intersection have?
11. S1 has 30 members. S2 has 28 members. Their union has 46 members. How many members does their intersection have?
12. S1 has 100 members. S2 has 108 members. Their union has 200 members. How many members does their intersection have?
13. S1 has 100 members. S2 has 128 members. Their union has 128 members. How many members does their intersection have?
14. Look for patterns in the following table.

Members in S1	Members in S2	Members in intersection	Members in union
8	8	4	12
8	8	3	13
8	8	2	14
8	9	4	13
8	10	4	14
8	11	4	15
9	11	5	15

15. Reflect on the things you learnt in this exploration. Write some things you learnt.

SOLUTIONS

1. S1 = {2, 4} S2 = {1, 4} S3 = {1, 4} S4 = {1, 3} S5 = {2, 3}.
2. The intersection of S1 and S2 = {4}.
3. The intersection of S1, S2 and S3 = {4}.
4. That square will have 4.
5. One way to represent set intersections is with the diagram shown below. This diagram is called a *Venn diagram*.

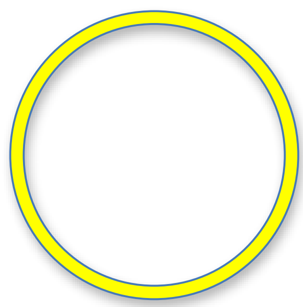


6. {1, 2, 3}
7. Represent information in a table like the one below or in a Venn diagram. The answer turns out that 14 are exclusively in F.

		Only F	Only R	Only C	F, R, not C	F, C not R	C, R not F	F, R, C
			16	10				6
60	All	Y	Y	Y	Y	Y	Y	Y
32	F	Y			Y	Y		Y
32	R		Y		Y		Y	Y
22	C			Y		Y	Y	Y
10	F, C					Y		Y

8. S1 = {1, 3} S2= {1, 2, 4}. The intersection of S1 and S2 (S1 ∩ S2) = {1}. Therefore, the value in the second square in the first column is 1.
9. {3, 4}
10. 2
11. 12
12. 8
13. 100
14. There are many different patterns in the table. One pattern is that the number of elements in S1 + the number of elements in S2 = the number of elements in the intersection + the number of elements in the union.

Draw a smiley in the circle below to celebrate your progress on this exploration.



Exploration of Divisibility

	1	2	3	4	5	6	7	8	9
A	5040x						18+		

Determining numbers that go in a KenKen® product cage like 5040x above involves finding if the target number is divisible by a particular factor. Divisibility rules are particularly helpful in doing this. In this exploration, we will study divisibility concepts.

EXERCISES

1) Examples of numbers divisible by 5 are: 5, 10, 15, 20, 15, 20, 105, 110, 205, 2300. Do you see any patterns in these numbers?

2) Do you see any patterns in the table below that lists numbers that are divisible by 9?

Number	27	927	9000	9909	20,007	17,127	900,009
Sum of digits	9	18	9	27	9	18	18

3) Do you see any patterns in the table below that lists numbers that are divisible by 6?

Number	222	1002	7008	2004	220,026
Sum of digits	6	3	15	6	12
Last digit	2	2	8	4	6

4) Do you see any patterns in the table below that lists numbers that are divisible by 11?

Number	22	1331	123,244	5060	7260
Sum of odd numbered digits	2	4	8	0	2
Sum of even numbered digits	2	4	8	11	13

5) You have numbers from 1 to 11, but not necessarily in that order. The product of the first nine is 441,760. The sum of the last two is 19. What are

the last two numbers?

6) KenKen® product cages in a puzzle have targets of 35, 80, 99, 96 and 100. Which of these are divisible by 5? Which of these are divisible by 6? Which of these are divisible by 9? Which of these are divisible by 11?

7) The following numbers are divisible by 11: A343, B15060, C22701, D030. What A, B, C and D?

8) 2313E is divisible by 6. What is E?

9) A number is divisible by 8 if the number formed by the last three digits is divisible by 8. Which of the following numbers are divisible by 8: (a) 12,001, (b) 24,007, (c) 11,022, (d) 456,008, (e) 456,012.

10) If you double the last digit of a number and subtract it from the rest of the numbers and the answer is 0, or divisible by 7, then the number is divisible by 7. Use this rule to determine which of the following numbers are divisible by 7: (a) 842, (b) 231, (c) 7078.

11) Reflect on the things you learnt in this exploration. Write some things you learnt.

SOLUTIONS

1) Numbers divisible by 5 end in 5 or 0.

2) The sum of the digits is a multiple of 9.

3) The sum of the digits is a multiple of 3 and the numbers end in an even number.

4) The difference between the sum of even numbered digits and odd numbered digits is a multiple of 11 or 0.

5) Looking at the sum of the last two numbers to be 19, we may conclude that 11 + 8 and 10 + 9 are two possibilities that would result in a sum of 19. We know that the product of the remaining numbers is 441,760. Let us examine if 11 is a factor of 441,760.

Obviously, we can try to divide 441,760 by 11. There is an easier way to determine if 11 is a factor of 441760. This involves using a rule called 'the divisibility rule of 11'.

Starting from the first digit, add all of the alternate digits to obtain Sum_odd.

Next, add the remaining digits to obtain Sum_even. Find the difference between Sum_odd and Sum_even (Sum_odd - Sum_even or Sum_even - Sum_odd). If the difference turns out to be 0 or a multiple of 11, then the original number will be divisible by 11.

Let us compute Sum_odd and Sum_even to see whether **441,760** is divisible by 11.

We obtain Sum_odd = 4 + 1 + 6 = 11 and Sum_even = 4 + 7 + 0 = 11.

Therefore, Sum_odd - Sum_even = 0. Hence, **441,760** is divisible by 11. If 11 is among the first 9 numbers, it must not be in the last two numbers. As we discussed earlier, the only two possibilities for the last two numbers are (11, 8) or (10, 9). Therefore, we can conclude that the numbers in the two-squares are 9 and 10.

So, the procedure for checking divisibility by 11 is:

1. Add the odd-numbered digits.
2. Add the even-numbered digits.
3. Subtract the smaller of the two sums from the larger of the two sums. If the number you obtain is divisible by 11, then so is the original number.

6) 35, 80 and 100 are divisible by 5. 96 is divisible by 6. 99 is divisible by 9. 99 is divisible by 11.

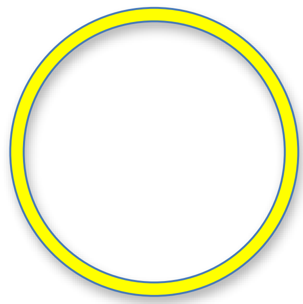
7) A343 is divisible by 11. Therefore, the difference between (A + 4) and 3 + 3 = 6 is a multiple of 11. If A + 4 = 6, then A = 2. With similar reasoning, we find that B is 1, C is 8 and D is 8.

8) 2313E is divisible by 6. Therefore, E is even. In addition, as the sum of the digits has to be divisible by 3, 9 + E is divisible by 3. Hence, E has to be 3, 6, or 9. As E is even, E has to be 6.

9) Only 456,008 is divisible by 8.

10) Only 231 is divisible by 7.

Draw a smiley in the circle below to celebrate your progress on this exploration.



Exploration of Multiplication Tricks

Reasoning about KenKen® product cages involves multiplication and division of a set of numbers. It is useful to learn a few tricks that allow us to multiply numbers quickly.

EXERCISES

1) What patterns do you observe in the table below?

Numbers	Product
2, 3, 5	30
2, 4, 5	40
2, 21 ,5	210
2, 28, 5	280
2, 18, 5	180

Now, try the following products using the pattern you observed.

2) 2 x 3 x 4 x 5

3) 2 x 3 x 4 x 5 x 6

4) 2 x 3 x 5 x 6

5) 2 x 2 x 3 x 5 x 5

6) What patterns do you observe in the table below?

Numbers	Product
4, 3, 25	300
4, 4, 25	400
4, 21, 25	2100
4, 28, 25	2800
4, 18, 25	1800

SOLUTIONS

1) To look for patterns, look for similarities between entries in each row and for similarities between different columns. We observe that the product is the same as the middle number followed by 0. In general, if we have a product of a series of numbers that includes 2 and 5, then do the following: (a) replace 2

and 5 by 10, (b) multiply the rest of the numbers (c) multiply the product by 10.

Because multiplying by 10 can be done easily by adding a zero at the end of the number, this re-ordering allows us to do the multiplication faster.

2) 120

3) 720

4) 180

5) 300

6) To look for patterns, look for similarities between entries in each row and for similarities between different columns. We observe that the product is the same as the middle number followed by 00. In general, if we have a product of a series of numbers that include 4 and 25, then do the following: (a) Replace 4 and 25 by 100. (b) Multiply the rest of the numbers (c) Multiply the product by 100. Because multiplying by 100 can be done easily by adding two zeros at the end of the number, this re-ordering allows us to do the multiplication faster.

Here are some more explorations regarding multiplication tricks:

A) Take a few even numbers and compare following results:
   a. The product you obtain after multiplying by 5
   b. The result you obtain by dividing the number by 2 and then multiplying by 10.

   Organize your results in a list. Are the results the same? Why? Which is an easier way to obtain to the answer?

B) Take a few numbers that are divisible by 4 and compare the following results:
   a. The product you obtain after multiplying by 25
   b. The result you obtain by dividing the number by 4 and then multiplying by 100.

   Organize your results in a list. Are the results the same? Why? Which is an easier way to obtain the answer?

C) Take a few numbers and compare the following results:
   a. The product you obtain after multiplying by 9
   b. The result you obtain by multiplying the numbers by 10 and then subtracting the original numbers.

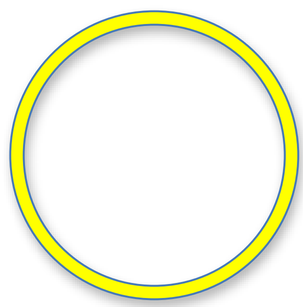
   Organize your results in a list. Are the results the same? Why? Which is an easier way to obtain to the answer?

D) Take a few numbers and compare the following results:
   a. The product you obtain after multiplying by 15
   b. The result you obtain by dividing the number by 2 and then multiplying the numbers by 30.

   Organize your results in a list. Are the results the same? Why? Which is an easier way to obtain the answer?

E) Reflect on the things you learnt in this exploration. Write some things you learnt.

Draw a smiley in the circle below to celebrate your progress on this exploration.



Exploration of Factor Pairs

Determining numbers that go in a two-square cage with a product target involves finding a factor pair, the product of which is the given target.

The procedure to find the factors of a given number is as follows:

1. Starting with 1, divide each number from 1 to the maximum allowed in the puzzle into the given number.
    a. If the numbers divide exactly and there is no remainder, then you have a pair of factors.
    b. List the divisor and the quotient of your division as a pair of factors.
2. Keep dividing until a factor pair repeats.

EXERCISES

1) Find the factor pairs of 12.

2) Find the factor pairs of 20.

3) Find the factor pairs of 25.

4) Find the factor pairs of 36.

5) Find the factor pairs of 49.

6) Find the factor pairs of 50.

7) Find numbers less than 100 that have odd number of factors.

8) Reflect on the things you learnt in this exploration. Write some things you learnt.

SOLUTIONS

1)

Number	Division	Factor Pair
1	12 / 1 = 12	1, 12
2	12 / 2 = 6	2, 6
3	12 / 3 = 4	3, 4
4	12 / 4 = 3	Repeat pair

Factor pairs of 12 are (1, 12), (2, 6), and (3, 4).

2)

Number	Division	Factor Pair
1	20 / 1 = 20	1, 20
2	20 / 2 = 10	2, 10
3	Not divisible	
4	20 / 4 = 5	4, 5
5	20 / 5 = 4	Repeat pair

The factor pairs of 20 are (1, 20), (2, 10) and (4, 5).

3) The factor pairs of 25 are (1, 25) and (5, 5).

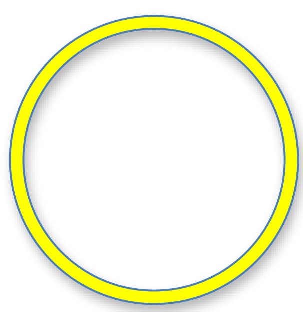
4) The factor pairs of 36 are (1, 36), (2, 18), (3, 12), (4, 9) and (6, 6).

5) The factor pairs of 49 are (1, 49) and (7, 7).

6) The factor pairs of 50 are (1, 50), (2, 25) and (5, 10).

7) Try numbers between 1 and 10. We find that the numbers with odd number of factors are 1, 4 and 9. Look for patterns. These are square numbers. This is so because square numbers have one factor pair where both numbers are the same and other factor pairs with two different numbers. Numbers less than 100 with odd number of factors are 1, 4, 9, 16, 25, 36, 49, 64, and 81.

Draw a smiley in the circle below to celebrate your progress on this exploration.



Exploration of Factor Triplets

Procedure to find factor triplets

1. First, find all factor pairs of a given number.
2. Now, for each factor pair, find factor pairs of the second factor. Replace this factor by the corresponding factor pairs.
3. Remove any duplicate factor triplets.

   For example:

   For 12, we have the following factor pairs (1) 1, 12 (2) 2, 6 (3) 3, 4

   When we replace 12 by its factor pairs, we will obtain:

   1, 1, 12

   1, 2, 6

   1, 3, 4

   When we replace 6 by its factor pairs, we will obtain:

   2, 1, 6

   2, 2, 3

   When we replace 4 by its factor pairs, we will obtain:

   3, 1, 4

   3, 2, 2

   We remove duplicate instances, what we obtain is:

   1, 1, 12

   1, 2, 6

   1, 3, 4

   2, 2, 3

EXERCISES

1) Find all factors of 15, 45, and 36.

To be sure that you found all of the factors, it would be useful to know how many factors a number has.

2) Number of factors of 1 is 1. Number of factors of 10 is 4. Number of factors of 100 is 9. Number of factors of 1000 is 16. Recognize the pattern. How many factors does 10,000 have?

3) Find all factor pairs and factor triplets of 15.

4) Find all factor triplets of 12, 16, and 18.

SOLUTIONS

1) Factors of 15 are 1, 3, 5, and 15. Factors of 45 are 1, 5, 3, 15, 9, and 45. Factors of 36 are 1, 2, 4, 3, 6, 12, 9, 18, and 36.

2) 10,000 has 25 factors.

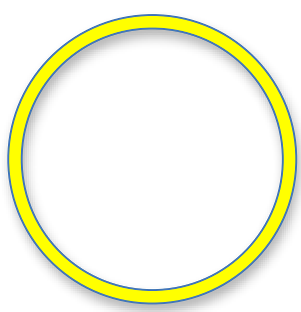
3) Factor pairs and factor triplets of 15 are (1, 15), (3, 5), (1, 1, 15), (1, 3, 5)

4) Factor triplets of 12 are (1, 1, 12), (1, 2, 6), (1, 3, 4), (2, 2, 3)

   Factor triplets of 16 are (1, 2, 8), (1, 1, 16), (1, 4, 4), (2, 2, 4)

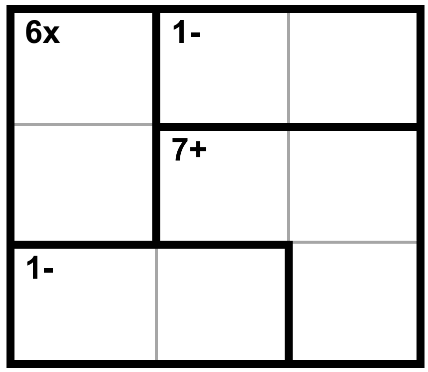
   Factor triplets of 18 are (1, 1, 18), (1, 2, 9), (1, 3, 6), (2, 3, 3)

   Draw a smiley in the circle below to celebrate your progress on this exploration.



Exploration of Creating a Table

Let us consider the following puzzle.



One may proceed to reason about this puzzle as follows. Numbers 2 and 3 would be the only ones that can go in 6x. The bottom left corner must have 1 as 2 and 3 go in the 6x cage in the same column. The bottom middle square must have 2 because it is in a cage with target 1- and the bottom left square has 1. The bottom right corner must have 3 as the bottom left square is 1 and the bottom middle square is 2. As bottom right square is 3, remaining squares in the 7+ cage must add up to 4. Therefore, the 7+ cage must have 3 and 1 as the middle and right square of the middle row. As 1, 2 and 3 must be in the middle row, 2 must be in the left square of the middle row. Now, we can conclude that the top row must have 3, 1 and 2 as left, middle and right squares of the top row.

Looking back and reflecting on a solution is often useful in obtaining further insights about a problem solving approach. In the above problem, it was helpful to know in the beginning that 6x cage has 2 and 3 even though we did not know the order. Often, it is useful to identify cages that have unique solutions. We would explore this further in this chapter.

Now, consider the following question:

Which two-square cages with target sums in a 20x20 KenKen® puzzle have unique solutions?

While you may be able to come up with one or two such cages, it is more difficult to be sure of all such cages. In general, if we have a difficult problem, it is useful to do exploration and obtain insights from explorations. Here, we will consider the following problem, which is a simpler version of the above-mentioned problem.

Which two-square cages with target sums in a 4x4 KenKen® puzzle have unique solutions?

A good strategy here is to make a table of the possible two-square cages.

To create a table, do the following:

1. Identify the related quantities.
2. Understand the relations between these quantities.
3. Understand the relation between the results and these quantities.
4. Understand the possible values these quantities can take and create an ordered list of these values.
5. Begin with the first value of one quantity and create all possible rows corresponding to this quantity by varying the values of other quantities from low to high.
6. Once, you are done with creating all possible rows corresponding to that possible value of the row, consider the next possible value of the quantity and create corresponding rows. Continue until you are done considering all possible values of this quantity.
7. For each of these rows in the table, compute results.

EXERCISES

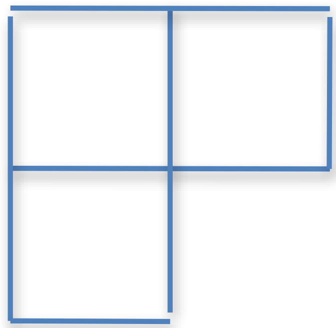
1) Create a table of all possible sums of two different numbers where each number can be 1, 2, 3, or 4. The first column in the table should be the smaller number and the second column should be the larger number.
2) In this table, identify sums for which there is a unique pair of numbers that can be added to obtain the target sum.
3) Create a table of all possible sums of two different numbers where each number can be 1, 2, 3, 4, or 5. The first column in the table should be the larger number and the second column should be the smaller number.
4) In the table you created in the previous question, identify sums for which there is a unique pair of numbers that can be added to obtain the target sum.
5) Create a table of all possible sums of two different numbers where each number can be 1, 2, 3, 4, 5, or 6. The first column in the table should be the larger number and the second column should be the smaller number.
6) In the table you created in the previous question, identify sums for which there is a unique pair of numbers that can be added to obtain the target sum.
7) Create a table of all possible sums of two different numbers where each number can be 1, 2, 3, 4, 5, 6, or 7. The first column in the table should be the larger number and the second column should be the smaller number.

8) In the table you created in the previous question, identify sums for which there is a unique pair of numbers that can be added to obtain the target sum.
9) Identify patterns in the following table.

Puzzle size	Two-square targets with unique solutions
4x4	2, 3, 6, 7
5x5	2, 3, 8, 9
6x6	2, 3, 10, 11
7x7	2, 3, 12, 13

10) What are the two-square target sums in a 10 x 10 KenKen® puzzle for which there are unique solutions?
11) What are the two-square target sums in a 11 x 11 KenKen® puzzle for which there are unique solutions?
12) What are the two-square target sums in a 12 x 12 KenKen® puzzle for which there are unique solutions?
13) What are the two-square target sums in a 13 x 13 KenKen® puzzle for which there are unique solutions?
14) What are the two-square target sums in a 14 x 14 KenKen® puzzle for which there are unique solutions?
15) Examine three-square row cages with target sums in 5x5, 6x6 and 7x7 KenKen® puzzles. Which cages have unique solutions?
16) Can you identify a pattern relating to the size of the KenKen® puzzle and the target sums for which there is a unique solution?
17) Can you identify the target sums in three-square row cages in 12x12 KenKen® puzzles for which we have unique solutions?

Now, consider three-square L-shaped puzzles like the one shown below and let us study cages with target sums for which there is a unique solution. In this cage, (1, 2) in the top row and 3 in the bottom square is regarded as different than (1, 3) in the top row and 2 in the bottom square.



18) Which target sums in an L-shaped three cage in a 4x4 KenKen® puzzle will have unique solutions?

19) Which target sums in an L-shaped three cage in a 5x5 KenKen® puzzle will have unique solutions?

20) Which target sums in an L-shaped three cage in a 6x6 KenKen® puzzle will have unique solutions?

21) Which target sums in an L-shaped three cage in a 7x7 KenKen® puzzle will have unique solutions?

22) What patterns can you identify in the following table that lists target sums in three-square L-shaped cages for which there are unique solutions?

Puzzle size	Targets with unique solutions
4x4	4, 5, 10, 11
5x5	4, 5, 13, 14
6x6	4, 5, 16, 17
7x7	4, 5, 19, 20

23) Find target sums in three-square L-shaped cages in 12x12 and 13x13 KenKen® puzzles for which there are unique solutions.

24) Identify patterns in the following product targets with unique solutions: 3, 5, 7, 11, 13.

25) Create a table of possible numbers to determine the minus target number in a 4x4 KenKen® for which there is a unique solution.

26) Create a table of possible numbers to determine the minus target number in a 5x5 KenKen® for which there is a unique solution.

27) Create a table of possible numbers to determine the minus target number in a 6x6 KenKen® for which there is a unique solution.

28) Can you identify a pattern relating to the size of the KenKen® puzzle and the minus targets in the puzzles for which there is a unique solution?

29) Create a table of possible numbers to determine the division target numbers in two-square cages in a 4x4 KenKen® for which there is a unique solution.

30) Create a table of possible numbers to determine the division target number in a 5x5 KenKen® for which there is a unique solution.

31) Create a table of possible numbers to determine the division target number in a 6x6 KenKen® for which there is a unique solution.

32) Can you identify a pattern relating to the size of a KenKen® puzzle and the division targets in the puzzles for which there is a unique solution?

33) What patterns do you see in the following table that lists product targets with unique solutions and associated puzzle size?

Size	**Product targets**
3	2, 3
4	2, 3
5	2, 3, 5
6	2, 3, 5
7	2, 3, 5, 7
11	2, 3, 5, 7, 11
14	3, 5, 7, 11, 13

34) Identify patterns in the following table relating product targets with unique solutions.

Size	Product Targets with Unique Solutions
3	6
4	6
5	6,10, 15
6	6, 10, 15
12	15, 21, 22, 26, 33, 35, 55, and 77

SOLUTIONS

1) Let us use the procedure described in this chapter to create a table. Our two columns can be the smaller of the two numbers and the larger of the two numbers. We know that the result of interest is the sum of these quantities. Possible values these numbers can take are 1, 2, 3, and 4. A list is [1, 2, 3, 4] We can start with the smaller number being 1 and consider the possibilities (1, 2), (1, 3) and (1, 4). Then, we can consider the possibility of the smaller number being 2. Here, we will consider the possibilities (2, 3) and (2, 4). Finally, we consider the smaller number being 3 and compute the sum to be 7. Now, we compute the sums of quantities in each row.

Smaller Number	Larger number	Sum
1	2	3
1	3	4
1	4	5
2	3	5
2	4	6
3	4	7

2) Examining the table for a 4x4 KenKen®, we can find that 3+, 4+, 6+ and 7+ are the targets with a single pair of numbers associated with targets.

3) Left for student to create the table.

4) Examining the table for a 5x5 KenKen®, we can find that 3+, 4+, 8+ and 9+ are the targets with a single pair of numbers associated with targets.

5) Left for student to create the table.

6) Examining the table for a 6x6 KenKen®, we can find that 3+, 4+, 10+ and 11+ are the targets with a single pair of numbers associated with targets.

7) Left for student to create the table.

8) Examining the table for a 7x7 KenKen®, we can find that 3+, 4+, 12+ and 13+ are the targets with a single pair of numbers associated with targets.

9) Let us examine patterns in the following table.

Puzzle size	Two-square targets with unique solutions
4x4	3, 4, 6, 7
5x5	3 ,4, 8, 9
6x6	3, 4,10, 11
7x7	3, 4, 12, 13

Do you see a pattern here in the numbers? **One strategy to look for patterns is to see what is common between successive rows.** First, 3 and 4 are common to all of these. Now, let us examine the largest target numbers in these rows: 7, 9, 11, 13.

When looking for a pattern, another strategy is to look for differences.

Differences between successive numbers here turn out to be two. Aha! So, these numbers are all increasing by two. There are two kinds of patterns: Patterns relating successive numbers and patterns relating entries in two columns. Another possible pattern between two types of quantities is that one is a multiple of another quantity. Here, we can try doubling numbers in the first column and we find that results are close to what we have in the second column, but fall short by one. What we notice is:

7 = 2 * 4 - 1 (* means ‘multiplied by’).

9 = 2* 5 - 1.

11 = 2 * 6 -1.

13 = 2 * 7 -1.

We can generalize this as the following hypothesis:

We can only create one pair of numbers that can create a sum of 2 * n - 1 in an n * n KenKen® puzzle.

Often, it is a good idea to verify a given hypothesis with additional examples. Here, if we can examine sums in 8x8 and 9x9 KenKen® puzzles, we find that this hypothesis is indeed true.

Now, let us examine the pair of numbers that result in the largest possible unique sum.

Puzzle size	Largest sum target with unique solution and associated numbers
4x4	7 = 3 + 4
5x5	9 = 4 + 5
6x6	11 = 5 + 6
7x7	13 = 6 + 7

Looking at the differences between successive rows, we will find that numbers increase by 1 from one row to the next row. Furthermore, the numbers are n and n - 1 for an n * n KenKen® puzzle. Now, let us see if we can create a logical explanation why a target 2n - 1 will have a unique solution. If one of the two numbers is less than (n - 1), then the other number will have to be larger than n for the sum to be 2n - 1. However, we can’t use a number larger than n in the sum. Therefore, we can’t have one of the numbers be smaller than (n - 1). In addition, we can’t use the numbers larger than n. Therefore, the only numbers we can use to create a sum of 2n - 1 are n and n -1.

Similarly, one can notice that there is a pattern between the second largest target number we have identified so far:

Puzzle size	Second largest target with unique solution
4x4	6
5x5	8
6x6	10
7x7	12

Can you identify a pattern here? Can you provide a logical explanation for the pattern you observe?

Again, when looking for a pattern, a possible strategy is to look for differences. Differences between successive numbers here turn out to be two. Therefore, these numbers are all increasing by two. Another pattern we notice is:

7 = 2 * 4 - 2.

9 = 2* 5 - 2.

11 = 2 * 6 - 2.

13 = 2 * 7 - 2.

Our observation here is the following:

We can only create one pair of numbers that can create a sum of 2 * n - 2 in an n * n KenKen® puzzle. Overall, the target sums n * n KenKen® puzzle with unique solutions are 3, 4, 2n - 2 and 2n -1.

10) 3, 4, 18, 19 are target sums in a 10 x 10 KenKen® puzzle where there are unique solutions.

11) 3, 4, 20, 21 are target sums in an 11 x 11 KenKen® puzzle where there are unique solutions.

12) 3, 4, 22, 23 are target sums in a 12 x 12 KenKen® puzzle where there are unique solutions.

13) 3, 4, 24, 25 are target sums in a 13 x 13 KenKen® puzzle where there are unique solutions.

14) 3, 4, 26, 27 are target sums in a 14 x 14 KenKen® puzzle where there are unique solutions.

15) In a 5x5 KenKen®, targets in three-square linear row cages with unique sums are 6, 7, 11, and 12. Those in a 6x6 KenKen® are 6, 7, 14, and 15. Those in a 7x7 KenKen® are 6, 7, 17, and 18.

16) The pattern: the unique target sums are 6, 7, and 3n - 4 and 3n - 3.

17) The targets in a 12x12 KenKen® with unique sums are 6, 7, 32, and 33.

18) 4, 5, 10, 11

19) 4, 5, 13, 14

20) 4, 5, 16, 17

21) 4, 5, 19, 20

22) An n x n KenKen® puzzle with three-square T-targets of 4, 5, and 3n - 1, 3n - 2 will have unique solutions.

23) 4, 5, 34, 35 are the target sums in three-square T-cages that have unique solutions in a 12x12 KenKen®. 4, 5, 37, 38 are the target sums in three-square T-cages that have unique solutions in a 13x13 KenKen®.

24) Those are prime numbers.

25) In a 4x4 KenKen®, the target is 3.

26) In a 5x5 KenKen®, the target is 4.

27) In a 6x6 KenKen®, the target is 5.

28) In an n x n KenKen®, the target is n - 1.

29) 4 and 3 are division target numbers in a 4x4 KenKen® for which there is a unique solution.

30) 5, 4, and 3 are division target numbers in a 5x5 KenKen® for which there is a unique solution.

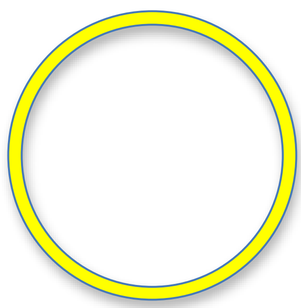
31) 6, 5, and 4 are division target numbers in a 6x6 KenKen® for which there is a unique solution.

32) For an n x n KenKen® puzzle, all numbers higher than n/2 and less than or equal to n are division target numbers for which there is a unique solution.

33) The second columns are all prime numbers smaller than the size of the KenKen® puzzle.

34) These numbers are higher than the maximum number allowed in any cage and these are products of two prime numbers.

Draw a smiley in the circle below to celebrate your progress on this exploration.

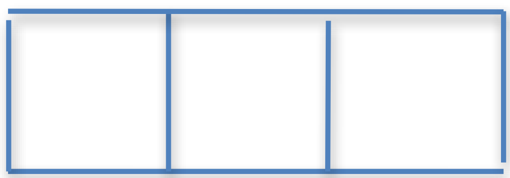


Exploration of Logical Reasoning

Given a situation, one can use logical reasoning to identify the following: What must be true? What must not be true? What may or may not be true?

Now, consider the following problem:

A three-square row cage is a cage of the kind shown below.



A three-square row cage in a 6x6 KenKen® puzzle has 13+ as a target. Can you say which numbers must be in the cage and which numbers must not be there?

Here, the numbers 2, 3, 4, and 5 may or may not be there. For example:

6 + 4 + 3 = 12.

6 + 5 + 2 = 12.

We have an example of a target sum with 5 as one of the numbers in the cage and we have an example of a target sum without 5 as one of the numbers. Therefore, 5 may or may not be there. Now, we will discuss one possible method that allows us to conclude that some number must be there or some number must not be there.

Example 1 of Proof by contradiction

We claim that 6 must be there.

To prove this, we will show what happens if we assume that 6 is not there.

If 6 is not there, the maximum sum possible with three numbers is 5 + 4 + 3 = 12.

However, the target sum is larger than 12.

Therefore, we have an impossible situation.

What we assumed must be false.

Therefore, 6 must be there.

Example 2 of Proof by contradiction

We claim that 1 must not be there.

To prove this, we will show what happens if we assume that 1 is there.

If 1 is there, the largest numbers among the remaining numbers would be 6 and 5. Then, the maximum sum possible with three numbers is 6 + 5 + 1 = 12.

However, the target is larger than 12.

Therefore, we have an impossible situation.

What we assumed must be false.

Therefore, 1 must not be there.

<u>The general procedure for proof by contradiction</u>

We want to show that X is true.

To show this, we will show what happens if we assume that X is not true.

If X is not true, logical reasoning would lead to an impossible situation.

Therefore, what we assumed must be false.

Therefore, X must be true.

This method of reasoning is called "proof by contradiction."

EXERCISES

Consider the following challenge problem:

We have a five-square row cage in a 12 x 12 KenKen® with a target sum of 17. What can we conclude about possible numbers in the cage? Which numbers must be there? Which numbers must not be there?

We will do an exploratory study to understand these types of problems so that we can progress on this problem.

1) What can you say about possible numbers in a three-square row cage in 6x6 KenKen® for target sums from 6 to 15?

2) For a three square row cage target sum of 8 in a 20x20 KenKen® puzzle, what is the largest possible number?

3) For a three square row cage a target sum of 49 in a 20x20 KenKen®, what is the smallest possible number?

4) For a four-square row cage with a target sum 35 in an 11x11 KenKen®, what is the smallest possible number?

5) For a four-square row cage with a target sum 11 in a 20x20 KenKen®, what is the largest possible number?

6) What can you say about possible numbers in a three-square row cage in a 12x12 KenKen® for target sums of: 6, 7, 8, 9, 10, 29, 30, 31, 32, 33? Which numbers must be there? Which number can't be there? Which numbers may be there?

7) Identify patterns in the table below.

Numbers allowed	The maximum sum possible with three different numbers
1, 2, 3, 4	9
1, 2, 3, 4, 5	12
1, 2, 3, 4, 5, 6	15
1, 2, 3, 4, 5, 6, 7	18
1, 2, 3, 4, 5, 6, 7, 8	21

8) Can you tell what is the maximum sum possible with three different numbers if we can use any number from 1 to 100?

9) Identify patterns in the table below.

Numbers allowed	Minimum sum possible with three different numbers
3 and higher	12
4 and higher	15
5 and higher	18
6 and higher	21
7 and higher	24

10) What is the minimum possible sum with three different numbers that are 11 or higher?

11) What is the minimum possible sum with three different numbers that are 20 or higher?

12) What is the minimum possible sum with three different numbers that are 100 or higher?

13) Can you determine the largest possible number in a three-square row cage in a 10x10 KenKen® with a target 8?

14) Can you determine the largest possible number in a three-square row cage in a 10x10 KenKen® with a target 9?

15) Find patterns in following table.

Target sum of a 3 square row cage	10	11	12	13
Maximum possible number	7	8	9	10

16) What is the smallest possible number in a three cage in a 10x10 KenKen® if the target sum is 22?

17) What is the smallest possible number in a three cage in a 10x10 KenKen® if the target sum is 23?

18) What is the smallest possible number in a three cage in a 10x10 KenKen® if the target sum is 24?

19) What is the smallest possible number in a three cage in a 10x10 KenKen® if the target sum is 25?

20) Can you recognize patterns in the table below:

Target sum for a three square linear cage	26	27	28	29
Smallest possible number for a 10x10 KenKen®	7	8	9	10

21) Identify patterns in the table below for a target sum of 29 in a three-square linear cage.

Size of puzzle	13x13	12x12	11x11
Smallest possible number	4	6	8

22) What is the smallest possible number with a target sum of 29 in a 14x14 KenKen®?

23) For a four-square linear cage with a target sum of 11, what is the largest possible number? Give your reasoning.

24) For a four-square linear cage with a target sum of 12, what is the largest possible number? Give your reasoning.

25) For a four-square linear cage with a target sum of 13, what is the largest possible number? Give your reasoning.

26) Identify patterns in the table below:

Target sum in linear four-square cage	14	15	16	17	18
Largest possible number	8	9	10	11	12

27) For a four-square row cage with a target sum of 34 in an 11x11, what is the smallest possible number?

28) For a four-square row cage with a target sum of 35 in an 11x11, what is the smallest possible number?

29) For a four-square row cage with a target sum of 36 in an 11x11, what is the smallest possible number?

30) For a four-square row cage with a target sum of 37 in an 11x11, what is the smallest possible number?

31) For a four-square row cage with a target sum of 34 in a 12x12, what is the smallest possible number?

32) For a four-square row cage with target sum 35 in a 12x12, what is the smallest possible number?

33) For a four-square row cage with target sum 36 in a 12x12, what is the smallest possible number?

34) For a four-square row cage with target sum 37 in a 12x12, what is the smallest possible number?

35) Identify patterns in the following table.

Size	12	12	12	13	13	13	13	13	14	14	14
Target	38	39	40	38	39	40	41	42	40	41	42
Smallest possible number	5	6	7	2	3	4	5	6	1	2	3

36) Reflect on the things you learnt in this exploration. Write some things you learnt.

SOLUTIONS

(1)

Sum targetof three-square linear cage in 6x6 KenKen®	Must be there	Must not be there	May be there
6	1, 2, 3	4, 5, 6	
7	1, 2, 4	3, 5, 6	
8	1	6	2, 3, 4, 5
9			1, 2, 3, 4, 5, 6
10			1, 2, 3, 4, 5, 6
11			1, 2, 3, 4, 5, 6
12			1, 2, 3 ,4, 5, 6
13	6	1	2, 3, 4, 5
14	3, 5, 6	1, 2, 4	
15	4, 5, 6	1, 2, 3	

2) For a three cage target sum of 8 in a 20x20 KenKen® puzzle, 5 is the largest possible number.

3) For a three-cage target sum of 49 in a 20x20 KenKen® puzzle, 10 is the smallest possible number.

4) For a four-square row cage with a target sum of 35 in an 11x11 KenKen® puzzle, 5 is the smallest possible number.

5) For a four-square row cage with target sum of 11 in a 20x20 KenKen®, 5 is the largest possible number.

6)

Sum target of three-square linear cage in 12x12 KenKen®	Must be there	Must not be there	May be there
6	1, 2, 3	4 and higher	
7	1, 2, 4	3, 5 and higher	
8	1	6 and higher	2, 3, 4, 5
9		7 and higher	1, 2, 3, 4, 5, 6
10		8 and higher	1, 2, 3, 4, 5, 6, 7
29		5 or smaller	6 or higher
30		6 or smaller	7 or higher
31	12	7 or smaller	8, 9, 10, 11
32	9, 11, 12	10, 8 or smaller	
33	10,11,12	9 or smaller	

7) Here are some strategies to look for differences:

Compare successive rows and look for similarities.

Compare successive rows and look for differences.

Look at differences between successive numbers and look for patterns in the differences.

Patterns in the table include the following:

- The sums are all multiples of three.
- The numbers allowed increase by one more number from one row to the next row.
- The sums increase by three in each successive row.
- The maximum sum is the sum of the largest three numbers.
- The maximum sum is three x (the largest number - 3).
- The maximum sum is three times the second largest number.

8) Given the pattern we observed in the previous question's analysis, we would expect that the maximum possible sum with the numbers from 1 to 100 would be 297. The reason for that is the largest possible sum would be the largest three numbers, which would be 98, 99, and 100. Therefore, this sum would be 98 + 99 + 100 = 297.

9) Here are some strategies to look for differences:

Compare successive rows and look for similarities.

Compare successive rows and look for differences.

Look at differences between successive numbers and look for patterns in differences.

Patterns in the table include the following:

The sums are all multiples of three.

The numbers allowed increase by one more number from one row to the next row.

The sums increase by three in each successive row.

The minimum sum is the sum of the smallest three numbers.

The minimum sum is three times the smallest number + 3.

The minimum sum is three times the second smallest number.

10) 36 is the minimum possible sum with three different numbers that are 11 or higher.

11) 63 is the minimum possible sum with three different numbers that are 20 or higher.

12) 303 is the minimum possible sum with three different numbers that are 100 or higher.

13) 5 is the largest possible number in a three-square row cage in a 10x10 KenKen® with target 8.

14) 6 is the largest possible number in a three-square row cage in a 10x10 KenKen® with target 9.

15) Patterns include the following:

Numbers in successive columns increase by one.

One strategy is to look at the differences between the values of two columns. Here, we find that the difference is always three. Therefore, another relevant pattern is the following:

The maximum possible number is the target sum minus three.

A way to explain this pattern is as follows. When we use the maximum possible number, the sum of the remaining two numbers would be the smallest possible sum. The minimum sum we can create with two different numbers would be the sum of 1 and 2, which is 3. Therefore, the maximum possible number that can be used in the cage is a target sum of minus three.

16) 3 is the smallest possible number in a three-cage in a 10x10 KenKen® if the target sum is 22.

17) 4 is the smallest possible number in a three-cage in a 10x10 KenKen® if the target sum is 23.

18) 5 is the smallest possible number in a three-cage in a 10x10 KenKen® if the target sum is 24.

19) 6 is the smallest possible number in a three-cage in a 10x10 KenKen® if the target sum is 25.

20) Pattern: The smallest possible number = Target sum - 19

21) Pattern:

The size of the puzzle reduces by 1 in successive columns.

The minimum number increases by 1 in successive columns.

For a three-cage target sum of x in an m * m KenKen®, the smallest possible number is x - m - (m - 1) if this number is positive.

22) Using the pattern above, we would conclude that it is 2.

23) For a four-square linear cage with a target sum of 11, 5 is the largest possible number. The largest possible number would correspond to the smallest sum created by the remaining three numbers. The smallest sum three numbers can create would be 1 + 2 + 3 = 6. Hence, the largest possible number would be 11 - 6 = 5.

24) For a four-square linear cage with a target sum of 12, 6 is the largest possible number. Give your reasoning. The largest possible number would correspond to the smallest sum created by the remaining three numbers. The smallest sum that three numbers would create would be 1 + 2 + 3 = 6. Hence, the largest possible number would be 12 - 6 = 6.

25) For a four-square linear cage with a target sum of 13, 7 is the largest possible number. Give your reasoning. The largest possible number would correspond to the smallest sum created by the remaining three numbers. The smallest sum three numbers would create would be 1 + 2 + 3 = 6. Hence, the largest possible number would be 13 - 6 = 7.

26) Identify patterns in the table below:

Target sum in linear four-square cage	14	15	16	17	18
Largest possible number	8	9	10	11	12

The largest possible number is T - 6 where T is the target sum in a linear four-square cage. This pattern can be explained as follows. The largest possible number would correspond to the smallest sum created by the remaining three numbers. The smallest sum three numbers can create would be 1 + 2 + 3 = 6. Hence, the largest possible number would be T - 6.

27) For a four-square row cage with a target sum of 34 in an 11x11, 4 is the smallest possible number.

28) For a four-square row cage with a target sum of 35 in an 11x11, 5 is the smallest possible number.

29) For a four-square row cage with a target sum of 36 in an 11x11, 6 is the smallest possible number.

30) For a four-square row cage with a target sum of 37 in an 11x11, 7 is the smallest possible number.

31) For a four-square row cage with a target sum of 34 in a 12x12, 1 is the smallest possible number.

32) For a four-square row cage with a target sum of 35 in a 12x12, 2 is the smallest possible number.

33) For a four-square row cage with a target sum of 36 in a 12x12, 3 is the smallest possible number.

34) For a four-square row cage with a target sum of 37 in a 12x12, 4 is the smallest possible number.

35) There are various patterns in the table:

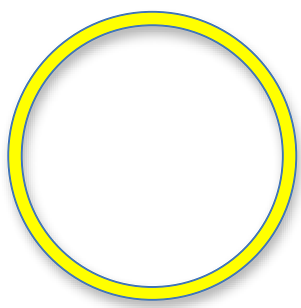
The smallest possible number = target - 3 * (size -1)

Size	12	12	12	13	13	13	13	13	14	14	14
Target	38	39	40	38	39	40	41	42	40	41	42
Smallest possible number	5	6	7	2	3	4	5	6	1	2	3

This can also be put in the following form:

The smallest possible number is T - L if L is the largest sum that can be made with three numbers.

Draw a smiley in the circle below to celebrate your progress on this exploration.

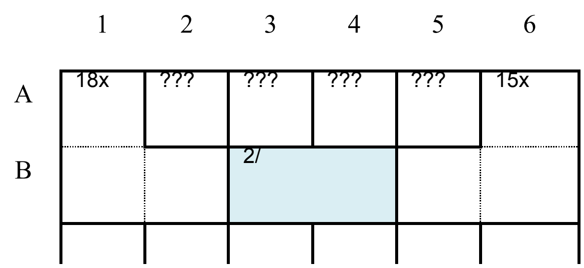


Exploration of Logic Charts

EXERCISES

1) Max, Yao and Naz are sitting in seats A, B, and C. Max does not sit in seat B. Yao sits in seat A. Where does everyone sit?

2) Consider the two rows in a 6x6 KenKen® puzzle that are listed below. Can you determine which numbers will go in the cage with the target is 2/.



SOLUTIONS

1) For certain type of puzzles, a graphical representation helps in understanding the description of clues. In the puzzle described in question 1, a representation that captures all associations between people and seats is helpful. This type of representation is shown below. This is called a logic grid.

	seat A	seat B	seat C
Max			
Yao			
Naz			

Now let us examine how we interpret each clue using such a logic chart. We have been told that Mr. Yao sits in seat A. As you read each clue, cross out the boxes that are not consistent with the clue. For instance, a sentence that tells you Mr. Yao sits in A allows you to determine he doesn't sit in seats B or C.

	seat A	seat B	seat C
Max			
Yao	Yes	X	X
Naz			

Once you've determined that Mr. Yao sits in seat A; then you can eliminate the possibility of sitting in seat A for any other person in the problem. As you can see, this really narrows down the options and helps you work toward a solution.

Thus, the chart would like this.

	seat A	seat B	seat C
Max	X		
Yao	Yes	X	X
Naz	X		

Now, let us represent the second clue in this table. Max does not sit in seat B.

	seat A	seat B	seat C
Max	X	X	
Yao	Yes	X	X
Naz	X		

If we know that Max sits in one of three seats and we have inferred that he can't be sitting in two of the three seats, then Max must be sitting in the remaining seat. So, in this case, we can infer that Max is sitting in seat C.

	seat A	seat B	seat C
Max	X	X	Yes
Yao	Yes	X	X
Naz	X		

If we know that Max is sitting in seat C, then Yao or Naz can't be sitting in seat C. Let's add this information to our logic chart.

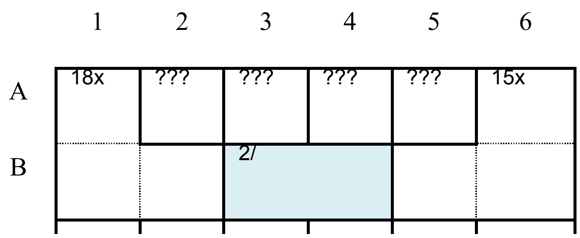
	seat A	seat B	seat C
Max	X	X	Yes
Yao	Yes	X	X
Naz	X		X

If we know that Yao sits in one of three seats and we have inferred that he can't be sitting in two of the three seats, then Naz must be sitting in the remaining seat. Therefore, in this case, we can infer that Naz is sitting in seat B. Now, let us add this information to our logic chart.

	seat A	seat B	seat C
Max	X	X	Yes
Yao	Yes	X	X
Naz	X	Yes	X

Now, we have filled all of the entries in our logic chart with a 'Yes' or an 'X'. Therefore, we have solved the puzzle. We know that Max sits in seat C. Yao sits in seat A and Naz sits in seat B.

2) Let us examine the pairs of numbers between 1 and 6 that satisfy the constraint 2/.



We find that the pairs (6, 3), (4, 2) and (2, 1) can all satisfy the constraint 2/. Thus, any of the numbers, 1, 2, 3, 4, or 6 can be the third or fourth square in the bottom row. Now, let us examine which triples can satisfy the constraint 18x. We find that <3, 2, 3> and <6, 3, 1> can satisfy the constraint 18x. Thus, any of the numbers 1, 2, 3, or 6 can be the first or second square of the bottom row. Similarly, <5, 3, 1> can satisfy the constrain 15x. Thus, the numbers, 1, 3, 5 can potentially be in the fifth or sixth square in the bottom row.

Here are our clues for the logic puzzle:

Cage	1	2	3	4	5	6
18x	possible	possible	possible			possible
2/	possible	possible	possible	possible		possible
15x	possible		possible		possible	

As 4 must be in one of the squares and only the 2/ cage lists it as a possibility, we can conclude that 4 must be the 2/ cage. Hence, the 2/ cage must have 2 and 4 as numbers. We can now revise the logic chart as below.

Cage	1	2	3	4	5	6
18x	possible	possible	possible			possible
2/		possible		possible		
15x	possible		possible		possible	

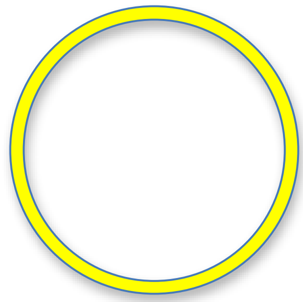
As 6 must be in one of the squares and only the 18x cage lists it as a possibility, we can conclude that 6 must be in the 18x cage. Therefore, other numbers in 18x must be 6, 3 and 1. We can now revise the logic chart as below.

Cage	1	2	3	4	5	6
18x	possible		possible			possible
2/		possible		possible		
15x	possible		possible		possible	

From this, we can conclude the following:

The 2/ cage has 2 and 4. The 18x cage has 1, 3 and 6. The 15x cage has 1, 3 and 5.

Draw a smiley in the circle below to celebrate your progress on this exploration.



Exploration of Sum and Difference

Consider the following problem:

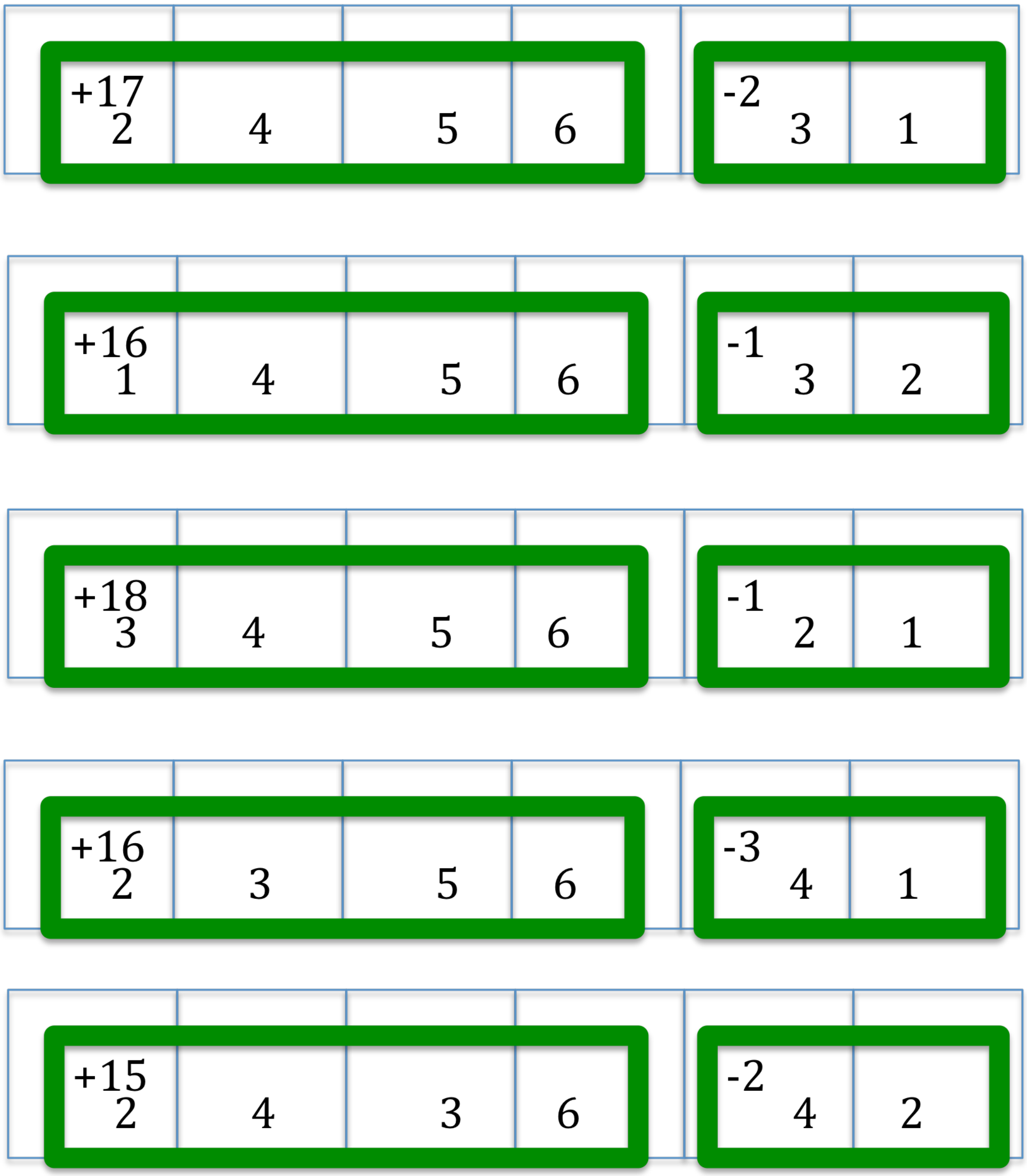
We have a list of ten numbers. In this list, the numbers 1 to 10 each occur exactly once, but not necessarily in the increasing order. The first eight numbers add up to 45. The difference between the last two is 2. What are the last two numbers?

A strategy that most students try to solve this problem is ‘guess and check’. For example, they may come up with a combination such as (2, 3, 4, 5, 6, 7, 8) and find that the difference between the remaining two numbers is not 2. It may take a long time to come up with the solution to this problem using the guess and check strategy.

We may believe that we are stuck. We need some insights to be able to make progress on the problem. **A strategy in such situations is to simplify the problem or to consider a similar problem.**

We can consider understanding similar situations in 6x6 puzzles. Consider the following exercise.

Exercise 1: Look at specific examples in Figure 1 and identify any patterns in it.



Solution to exercise 1: There are various patterns one may observe. These examples including the following:

- As the fifth number increases by 1, the target sum for the first cage decreases by 1 as well.
- As the sixth number increases by 1, the target sum for the first cage decreases by 1 as well.
- The target sum of the first cage = 21 - a - b if a and b are the fifth and sixth numbers.
- The sum of the fifth and sixth numbers = 21 – the target sum of the first cage.
- When a number in the first cage increases by 1 and other numbers in that cage remain the same, a number in the second cage decreases by 1.
- When a number in the first cage increases by 1 and other numbers in that cage remain the same, the target sum for that cage increases by 1.
- If we increase a number in the first cage by 1, the target sum for the cage increases by 1.

Exercise 2: Can we explain why the sum of the fifth and sixth numbers equals the difference between 21 and the target sum of the first cage?

Solution to exercise 2: The pattern about the sum of the last two numbers allows inferring the sum of the last two numbers. Sometimes, a number relationship becomes clearer when we develop a mathematical model. Suppose the target for a sum cage is a and the sum of the rest of the numbers outside the cage is b. We also know that all of the numbers in the row are 1 to 6. Therefore, all numbers together will add up to 1 + 2 + 3 + 4 + 5 + 6 = 21.

Therefore, we can say:

a + b = 1 + 2 + 3 + 4 + 5 + 6

This is a mathematical model for reasoning about numbers in the row.

It can be manipulated mathematically.

For example, we can write

b = 21 - a.

Because we know the value of a, we can determine the value of b as well by solving this equation. If we know b, we know an additional constraint that the last two numbers would add up to b. In some problems, this would allow us to make further inferences. Now consider the following problem:

Suppose the last two numbers are x and y. Then the -2 target tells us that x - y = 2. Furthermore, based on what we discussed earlier, the last two numbers would add to 4. In addition, we have been told that the difference between them is 2.

This can be written as:

x - y = 2

x + y = 4.

Again, we have created a mathematical model here. Creating such a model would allow us to understand strategies to solve this type of problems that will recur in different forms in KenKen® puzzles.

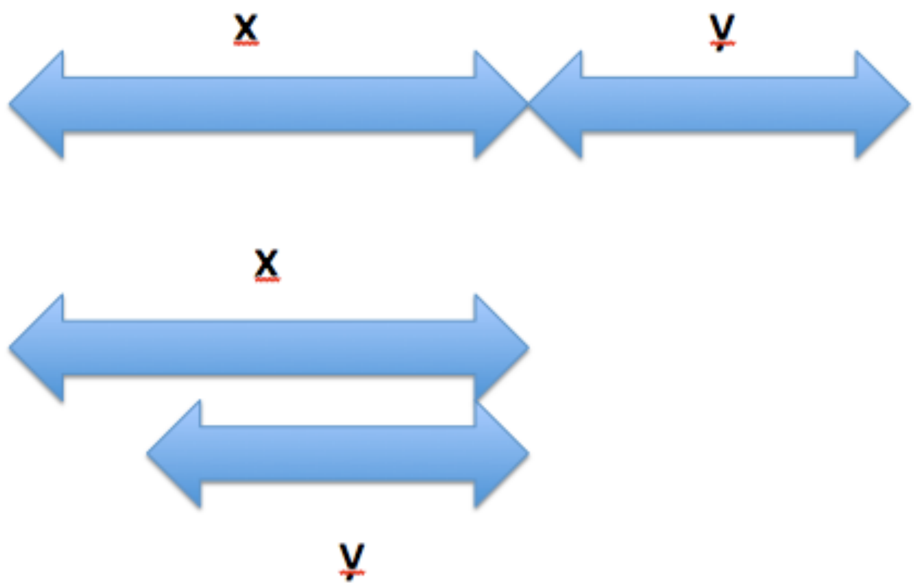
Now, we need a procedure to determine the two numbers.

As the problem involves two unknown variables, one strategy would be guess and check. In this case, you will probably obtain the answer after making a few guesses. As the sum of x and y is 4, each of these can, at most, be 4. One can create a table listing possible values of x and y and their corresponding sums. Part of such a table is shown below.

x	1	2	3	4	1	2	3
y	1	1	1	1	1	1	1
x + y	2	3	4	5	1	3	4
x- y	0	1	2	3	0	1	2

One can then identify values in the table for which the sum of the two numbers is 4 and the difference between the two numbers is 2. Matching numbers are x = 3 and y =1.

Now, let us examine additional approaches to solve this problem. Sometimes, when a solution to a problem is not obvious, we can represent the problem information in a diagram and then the solution becomes obvious. Below is one such representation.



Visually, we may be able to see that the difference between x + y and x - y is 2y.

Thus, 2y = 4 - 2.

Hence, 2y = 2.

Dividing both sides by two, we obtain:

y = 1.

Once, we have the value of y, we can substitute it in the equation x + y = 4

That will give us x + 1 = 4.

Now, subtracting 1 from both sides of the equation, we will obtain:

x + 1 - 1 = 4 - 1.

Hence, x = 3.

Below, we provide a simple three-step procedure to determine the numbers with a sum of a and a difference of b

(1) Let x and y be the two numbers. Calculate y to be (a - b) / 2.
(2) Substitute the value of y in the equation x + y = a.
(3) Simplify this equation to calculate the value of x.

This is because (x + y) + (x - y) = 2x = a + b.

Similarly, (x + y) - (x - y) = 2y = a – b.

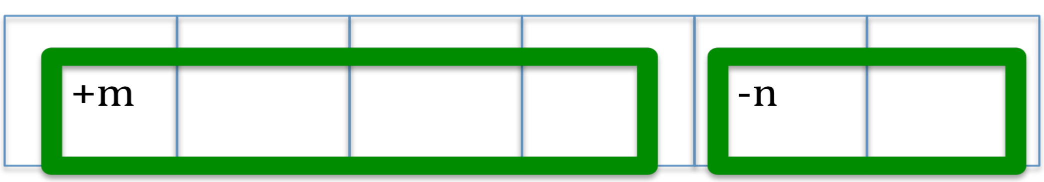
EXERCISES

For each of the problems in the table below, find x and y. Solve this problem using two methods: drawing a diagram and using the three-step procedure described in the chapter.

Problem Number	x + y	x -y	What is x?	What is y?
1	4	12		
2	6	20		
3	12	32		
4	100	140		
5	3	9		
6	17	19		
7	25	27		
8	25	77		
9	50	90		
10	220	300		

SOLUTIONS

Problem Number	x + y	x -y	What is x?	What is y?
1	4	12	8	4
2	6	20	13	7
3	12	32	22	10
4	100	140	120	20
5	3	9	6	3
6	17	19	18	1
7	25	27	26	1
8	25	77	51	26
9	50	90	70	20
10	220	300	260	40



The procedure for determining two rightmost numbers in a row of the kind above in a KenKen® puzzle where the rightmost cage has a subtraction target n consisting of two-squares and the remaining squares are in a cage with target sum m:

(1) Add all numbers allowed in the KenKen® puzzle.
(2) Calculate the sum of the two rightmost numbers to be the difference between the result in step (1) and m.
(3) The difference between the two rightmost numbers is n.
(4) Now, determine the numbers using the procedure described above for determining numbers the sum and difference of which is given.

Now consider the problem we discussed earlier:

I have a list of ten numbers. In this list, the numbers 1 to 10 each occur exactly once, but not necessarily in increasing order. The first eight numbers add up to 45. The difference between the last two is 2. What are the last two numbers?

Adding all of the numbers from 1 to 10, we obtain 55.

Because the first eight numbers add up to 45, the sum of the rightmost numbers will be 55 - 45 = 10

The difference between the rightmost two numbers is 2.

Hence, using the procedure for finding numbers from the sum and difference, we can determine the numbers to be 6 and 4.

Answer: 6 and 4

EXERCISES

I have a list of six numbers. The numbers are 1 to 6, but not necessarily in the same order.

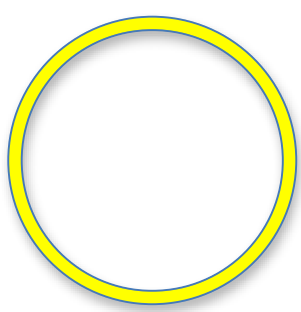
Each row in the table below specified the sum of the first four numbers and difference in the last two numbers, Try to determine the last two numbers. Each row has a different answer.

Sum of first four numbers	Difference in last two numbers
14	5
13	2
14	1
17	2
14	3
13	4
15	2
12	1
16	3
15	4
12	3

SOLUTIONS

Sum	Difference	Larger number	Smaller Number
14	5	6	1
13	2	5	3
14	1	4	3
17	2	3	1
14	3	5	2
13	4	6	2
15	2	4	2
12	1	5	4
16	3	4	1
15	4	5	1
12	3	6	3

Draw a smiley in the circle below to celebrate your progress on this exploration.



Exploration of Product and Difference

EXERCISES

1) Consider the following problem:

I have a list of eight numbers. In this list, the numbers 1 to 8 each occur exactly once, but not necessarily in increasing order. The product of the first six numbers is 720. The difference between the last two numbers is 1. What are the last two numbers?

Attempt to solve the problem with the guess and check strategy where you guess the numbers. If you obtain an answer, write it down. Try to identify the difficulties in using the guess and check strategy on this problem.

2) Each row in the table below lists properties of a different list of 6 numbers consisting of numbers from 1 to 6 in different orders. What patterns do you observe in the table?

First four numbers	Last two numbers	Product of first four	Product of last two	Difference between last two
1, 2, 3, 4	5, 6	24	30	1
1, 2, 3, 5	4, 6	30	24	2
1, 2, 4, 5	3, 6	40	18	3
1, 3, 4, 5	2, 6	60	12	4
2, 3, 4, 5	1, 6	120	6	5

3) Each row in the table below lists properties of a different list of six numbers consisting of numbers from 1 to 6 in different orders. What patterns do you observe in the table?

First four numbers	Last two numbers	Product of first four	Difference between last two
1,2 , 3, 4	5, 6	24	1
2, 3, 4, 5	1, 6	120	5
1, 3, 4, 5	2, 6	60	4
2, 3, 4, 5	1, 6	120	5
1, 2, 4, 5	3, 6	40	3
2, 3, 4, 5	1, 6	120	5

4) We have a list of six numbers from 1 to 6, but not necessarily in that order. The product of first four numbers is 24. We are trying to determine the product of the last two numbers. We know that the product of all six numbers = 1 x 2 x 3 x 4 x 5 x 6 = 720. Suppose the product of the last two numbers is p.

720 = the product of all six numbers

Therefore, 720 = the product of the first four numbers multiplied by the product of the last two numbers.

Therefore, we determine that 720 = 24 x p. From this, determine the value of p.

5) We have been given two mystery numbers x and y that can be 1, 2, 3, 4, 5 or 6. We have been told:

x - y = 1.

x * y = 30.

Can you determine x and y from this information? One possible method to do this is to create a table with possible values of x and y and check whether given constraints are true.

6) We are told that the difference between two numbers is four and the product of these two numbers is 32. We want to determine the sum of these two numbers.

   A. What is the difference between the two numbers?
   B. What is the square of the difference between the two numbers?
   C. What is 4 multiplied by the product of the two numbers?
   D. What is the sum of the results of (B) and (C)?
   E. What is the square root of (D)?
   F. What is the sum of the two numbers?

7) We are told that the difference between two numbers is 2 and the product of these two numbers is 8. We want to determine the sum of these two numbers.

   A. What is the difference between the two numbers?
   B. What is the square of the difference between the two numbers?
   C. What is 4 multiplied by the product of the two numbers?
   D. What is the sum of the results of (B) and (C)?
   E. What is the square root of (D)?
   F. What is the sum of the two numbers?

8) We are told that the difference between two numbers is 1 and the product of the two numbers is 30. We want to determine the sum of these two numbers.

   A. What is the difference between the two numbers?
   B. What is the square of the difference between the two numbers?
   C. What is 4 multiplied by the product of the two numbers?
   D. What is the sum of the results of (B) and (C)?
   E. What is the square root of (D)?
   F. What is the sum of the two numbers?

9) We are told that difference between two numbers is 3 and the product of these two numbers is 18. We want to determine the sum of these two numbers.

   A. What is the difference between the two numbers?
   B. What is the square of the difference between the two numbers?
   C. What is 4 multiplied by the product of the two numbers?
   D. What is the sum of the results of (B) and (C)?
   E. What is the square root of (D)?
   F. What is the sum of the two numbers?

10) We are told that the difference between two numbers is 1 and the product of these two numbers is 42. We want to determine the sum of these two numbers.

A. What is the difference between the two numbers?
B. What is the square of the difference between the two numbers?
C. What is 4 multiplied by the product of the two numbers?
D. What is sum of the results of (B) and (C)?
E. What is the square root of (D)?
F. What is the sum of the two numbers?

11) We are told that the difference between two numbers is 3 and the product of the two numbers is 4. We want to determine the sum of these two numbers.

A. What is the difference between the two numbers?
B. What is the square of the difference between two numbers?
C. What is 4 multiplied by the product of the two numbers?
D. What is the sum of the results of (B) and (C)?
E. What is the square root of (D)?
F. What is the sum of the two numbers?

12) We are told that the difference between two numbers is four and the product of these two numbers is 12. We want to determine the sum of these two numbers.

A. What is the difference between the two numbers?
B. What is the square of the difference between the two numbers?
C. What is 4 multiplied by the product of the two numbers?
D. What is the sum of the results of (B) and (C)?
E. What is the square root of (D)?
F. What is the sum of the two numbers?

13) If you are given the sum and difference between two positive numbers, then you can find the two numbers using the equations below:

Larger number = (sum + difference)/2

Smaller number = (sum - difference)/2

(a) The sum of two numbers is 12. Their difference is 4. What is the larger number? What is the smaller number?

(b) The sum of two numbers is 6. Their difference is 2. What is the larger number? What is the smaller number?

(c) The sum of two numbers is 11. The difference between the two numbers is 1. What is the larger number? What is the smaller number?

(d) The sum of two numbers is 9. The difference between the two numbers is 3. What is the larger number? What is the smaller number?

(e) The sum of two numbers is 13. The difference between the two numbers is 1. What is the larger number? What is the smaller number?

(f) The sum of two numbers is 5. The difference between the two numbers is 3. What is the larger number? What is the smaller number?

(g) The sum of two numbers is 8. The difference between two numbers is 4. What is the larger number? What is the smaller number?

14) For each of the problems in the table below, find x and y.

Problem Number	x * y	x -y	What is x?	What is y?
1	32	12		
2	8	2		
3	30	1		
4	18	3		
5	42	1		
6	4	3		
7	12	4		

15) Now, let us consider the first problem.

I have a list of 8 numbers. In this list, the numbers 1 to 8 each occur exactly once, but not necessarily in increasing order. The product of the first six numbers is 720. The difference between the last two numbers is 1. What are the last two numbers?

(a) What is the product of all of the numbers from 1 to 8?

(b) If 720 * a = 1 * 2 * 3 * 4 * 5 * 6 * 7 * 8, then what is a?

(c) If the difference between two numbers is 1 and their product is given by the answer to the previous question, then what is the sum of the two numbers?

(d) If the sum of the numbers is given by the answer to the previous question and the difference between the two numbers is 1, then what are the two numbers?

16) Can you determine what two numbers are in the first cage given the difference between the first two numbers and the product of the last four numbers in the next table? Each row corresponds to a different problem and has a different solution.

Difference between first two	Product of last four	Larger number	Smaller number
4	144		
2	30		
1	360		
1	120		
3	72		
3	40		
3	180		
1	60		
2	48		
5	120		
2	90		
1	36		
1	24		

17) Look back and identify some things you learnt from this exploration.

SOLUTIONS

1. A strategy that most students use on this problem is guess and check. While this strategy can lead to a solution, it is a time-consuming strategy that involves a lot of computation.

2. There are various patterns one may observe in these examples including the following:

   As the last two numbers become smaller, the product of the first four increases. The product of the last two numbers and the first four numbers is always 720.

3. There are various patterns one may observe in these examples including the following:

   If the fifth number is a and it is replaced by 1, the product of the first four is multiplied by a. The product of the last two numbers and the first four numbers is always 720.

4. p is 30.

5. x is 6 and y is 5.

6. (a) 12 (b) 16 (c) 128 (d) 144 (e) 12 (f) 12.

7. (a) 2 (b) 4 (c) 32 (d) 36 (e) 6 (f) 6.

8. (a) 1 (b) 1 (c) 120 (d) 121 (e) 11 (f) 11.

9. (a) 3 (b) 9 (c) 72 (d) 81 (e) 9 (f) 9.

10. (a) 1 (b) 1 (c) 168 (d) 169 (e) 13 (f) 13.

11. (a) 3 (b) 9 (c) 16 (d) 25 (e) 5 (f) 5.

12. (a) 4 (b) 16 (c) 48 (d) 64 (e) 8 (f) 8.

13. (a) 8 and 4 (b) 4 and 2 (c) 10 and 9 (d) 6 and 3 (e) 7 and 6 (f) 4 and 1 (g) 6 and 2.

14.

Problem Number	x * y	x - y	What is x?	What is y?
1	32	12	8	4
2	8	2	4	2
3	30	1	6	5
4	18	3	6	3
5	42	1	7	6
6	4	3	4	1
7	12	4	6	2

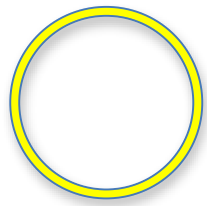
15. (a) 40320 (b) 56 (c) 15 (d) 7 and 8.

16.

Difference between first two	Product of last four	Larger number	Smaller number
4	144	5	1
2	30	6	4
1	360	2	1
1	120	3	2
3	72	5	2
3	40	6	3
3	180	4	1
1	60	4	3
2	48	5	3
5	120	6	1
2	90	4	2
1	36	5	4
1	24	6	5

17. Initially, you may be stuck. However, exploring similar but simpler problem situations can provide us insights that helps to solve the problem. Though guess and check is one possible strategy that can be used for this problem, we learnt another way to solve the problem involving sums and differences of numbers.

Draw a smiley in the circle below to celebrate your progress on this exploration.



Exploration of Sum and Division

EXERCISES

1) Consider the following problem:

I have a list of ten numbers. In this list, the numbers 1 to 10 each occur exactly once, but not necessarily in increasing order. The sum of the first eight numbers is 46. The ratio of the last two numbers is 2. What are the last two numbers?

Attempt to solve the problem with the guess and check strategy where you guess the numbers. If you obtain an answer, write it down. Try to identify the difficulties in using the guess and check strategy on this problem.

2) Solve the following problems:

a. What is the sum of the numbers from 1 to 10?

b. If the first eight numbers add to 46, what is the sum of the last two numbers?

3) Create a table where the first column values vary from 1 to 4, the second column's value is twice as much as the first column's value, the third column's value is the sum of the first two column's values, and the fourth column's value is the second column's value divided by the first column's value.

4) If the sum of two numbers is 9 and their ratio is 2, what are the two numbers?

5) Now, let us practice our skills on the following problems about determining unknown numbers from their sum and product.

Each row in the table below is a different problem. Determine the two unknown numbers in each case.

Problem Number	Sum of numbers	Ratio of numbers	Numbers
1	27	2	
2	54	2	
3	81	2	
4	60	2	
5	44	3	
6	60	3	
7	400	3	
8	60	4	

After observing your solutions to questions in the table, can you discover any patterns relating numbers with their sums and ratios?

6) The first row of a 10x10 KenKen® has two horizontal cages. The first is an eight-square cage with a target sum S. The second one is a two-square cage with a division target R. Try to determine the numbers.

Problem Number	S	R	Find numbers
1	45	4	
2	52	2	
3	43	2	
4	40	2	

7) Can you discover a pattern in the solutions to the previous problem?

8) The first row of a 9x9 KenKen® has two horizontal cages. The first is a seven-square cage with a target sum. The second is a two-square cage with a division target. Try to determine the numbers.

Problem Number	Sum	Ratio	Find numbers
1	41	3	
2	33	2	
3	43	3	
4	30	2	

9) Can you discover a pattern in the solutions to the previous problem?

SOLUTIONS

1) A strategy that most students use on this problem is guess and check. While this strategy can lead to a solution, it is a time-consuming strategy that involves a lot of computation. Because it involves a lot of computation, this strategy is error-prone as well.

2) The sum of the numbers from 1 to 10 is 55. If the last two numbers add to a, then we can write it as follows:

The sum of the first eight numbers + the sum of all pairs of numbers = the sum of all numbers that equal 55.

We know that the sum of the first eight numbers is 46.

So, $46 + a = 55$.

We can simplify this by subtracting 46 from both sides.

$46 + a - 46 = 55 - 46$.

$a + 0 = 9$.

Hence, a must be 9.

3)

First Number	Second Number	Sum	Ratio
1	2	3	2
2	4	6	2
3	6	9	2
4	8	12	2

4) Suppose, the smaller of the last two numbers is a. As the ratio of two numbers is 2, then the two numbers are a and 2a. As we know the sum of the two numbers is 9, we can write as follows:

$a + 2a = 9$.

To determine what a is, let us realize that a + 2a should be 3a. Then, we can write:

$3a = 9$.

So, $a = 3$.

As $a = 3$, the last two numbers are 3 and 6.

5)

Problem Number	Sum of numbers	Ratio of numbers	Numbers
1	27	2	9, 18
2	54	2	18, 36
3	81	2	27, 54
4	60	2	20, 40
5	44	3	11, 33
6	60	3	15, 45
7	400	3	100, 300
8	60	4	12, 48

Pattern: Smaller number = (Sum of numbers) / (1 + ratio of numbers)

6) Now, examine the solutions of these problems below.

Problem Number	Sum	Ratio	Find numbers
1	45	4	2, 8
2	52	2	1, 2
3	43	2	4, 8
4	40	2	5, 10

7) The smaller number = (55 - sum) / (1 + ratio)

8) 9x9 KenKen®

Problem Number	Sum	Ratio	Find numbers
1	41	3	1, 3
2	33	2	4, 8
3	33	3	3, 9
4	30	2	5, 10

9) The smaller number = (45 - sum) / (1 + ratio)

Below, we develop a formula to determine the numbers where the ratio of two numbers is r and the sum of two numbers is b.

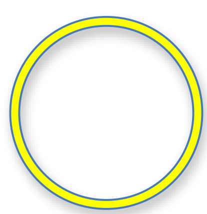
Let the two numbers be a and r * a.

We can write a + r * a = b.

Solving this, we will obtain a = b / (1 + r).

Now, we know one number to be a. The second number would be r * a.

Draw a smiley in the circle below to celebrate your progress on this exploration.



Exploration of Product and Quotient

EXERCISES

1. Given xy = 8 and x/y = 2, what is x and what is y?
2. Find x and y given the product and the ratio of x, y in the first two columns.

Product of x & y	Ratio of x & y	x	y
4	1		
9	1		
16	1		
8	2		
18	2		
3	3		
12	3		

3. Identify patterns in the table below.

Product of x & y	Ratio of x & y	x	y
4	1	2	2
9	1	3	3
16	1	4	4
8	2	4	2
18	2	6	3
3	3	3	1
12	3	6	2

SOLUTIONS

(1) As x/y = 2, x is double of y. We will create a table of possible values of x and y where x is double of y.

y	x	xy
1	2	2
2	4	8
3	6	18

Looking at the table, we can conclude that y = 2 and x = 4.

(2)

Product of x & y	Ratio of x & y	x	y
4	1	2	2
9	1	3	3
16	1	4	4
8	2	4	2
18	2	6	3
3	3	3	1
12	3	6	2

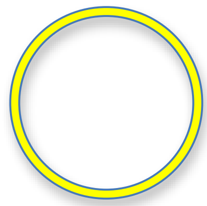
(3) Look for patterns in the table below

Product of x & y	Ratio of x & y	x	y
4	1	2	2
9	1	3	3
16	1	4	4
8	2	4	2
18	2	6	3
3	3	3	1
12	3	6	2

One may observe various patterns: As product increases and the ratio is the same, x increases and y increases.

The product of the ratio and the product is the square of x. The product divided by the ratio is the square of y.

Draw a smiley in the circle below to celebrate your progress on this exploration.



Exploration of Probability

The table below lists combinations of possible numbers that can go in a 2-cell cage.

Smaller Number	Larger number	Sum
1	2	3
1	3	4
1	4	5
2	3	5
2	4	6
3	4	7

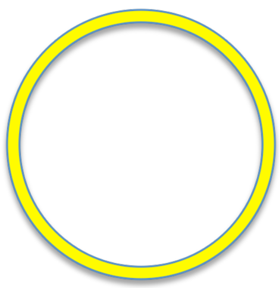
EXERCISES

1. If we randomly pick one of these combinations, what is the chance that the sum will be 3?
2. If we randomly pick one of these combinations, what is the chance that the sum will be 5?
3. How many different entries consisting would be a table listing a smaller number and a larger number where the numbers can be any number from 1 to 2?
4. How many different entries consisting would be a table listing a smaller number and a larger number where the numbers can be any number from 1 to 3?
5. How many different entries consisting would be a table listing a smaller number and a larger number where the numbers can be any number from 1 to 4?
6. How many different entries consisting would be a table listing a smaller number and a larger number where the numbers can be any number from 1 to 5?
7. Do you see any patterns in the answers to the previous question?
8. How many different entries consisting would be a table listing a smaller number and a larger number where the numbers can be any number from 1 to 11?

SOLUTIONS

1. 1/6
2. 1/3
3. 1
4. 3
5. 6
6. 10
7. You may notice any of various patterns. For example: Number of entries increases as number of possible values increases. Number of entries is equal to n * (n - 1) / 2
8. Number of entries is 11 * 10 / 2 = 55

Draw a smiley in the circle below to celebrate your progress on this exploration.



Exploration of Changing Representation

Sometimes, a problem as described seems hard, but when it is represented using different language or a diagram, it seems easier. Our use of diagrammatic representation is an example of this. Let us examine another example below:

We have eight numbers from 1 to 8, but not necessarily in that order. Product of first two is 8. Difference between the third and the fourth number is 5. Difference between the fifth number and the sixth number is 1. Difference between the seventh number and the eighth number is 1. What are the first two numbers?

A natural approach is to consider possible numbers that meet each of the clues given. However, this appears complex because of number of possibilities is large. Now, we will consider an idea that is based on the work of Prof Reiter.

Let us consider evenness and oddness of numbers and number of even or odd numbers in each cage. This is a different way of looking at the problem. We will count the number of even numbers between the third and forth numbers. As the difference is 5, one of the numbers has to be even. Now, let us count number of even numbers between the fifth and the sixth numbers. As the difference is 1, one of the numbers has to be even. Similarly, there has to be one even number between the seventh and the eighth numbers. Thus, we have counted three even numbers among the last six numbers. We know that there are 4 even numbers among all eight numbers as the numbers are 1, 2, 3, 4, 5, 6, 7 and 8, but not in that order. Now, let us post the following question:

We have eight numbers that consist of four even numbers and four odd numbers. Between the third and the fourth numbers, one is even and one is odd. Between the fifth and the sixth numbers, one is even and one is odd. Between the seventh and the eighth numbers, one is even and one is odd. How many even numbers are there between the first two numbers?

Clearly, there will be one even number and one odd number.

Now, let us examine the clue about the product of first two numbers being 8. If we know that only one of these is even, then the only possible pair of first two numbers is (1, 8). We can realize that by examine all possibilities of numbers with product of 8: (1, 8) and (2, 4).

Let us reflect on this problem solving experience. Using the idea of focusing of evenness and oddness of numbers was important in helping us solve this problem. In general, changing representation can sometimes be helpful in solving hard problems.

Now consider the following:

Suppose we are told that eight numbers add up to 100. What is the maximum number of times even numbers may occur among these eight numbers?

You may find this to be a hard problem. A good strategy to attack hard problem is do an exploratory analysis. Many of the exercises below involve such an exploration.

EXERCISES

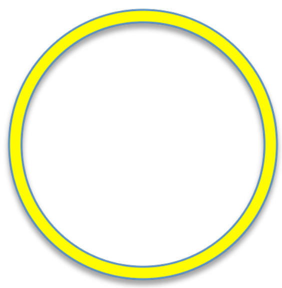
a. Consider many examples of two numbers where the difference between the two numbers is odd. Look for patterns. How many odd numbers are there?
b. Consider many examples of two numbers where the difference between the two numbers is even. Look for patterns. How many odd numbers are there?
c. Consider many examples of two numbers where the product of the two numbers is a multiple of 2, but not a multiple of 4. Look for patterns. How many odd numbers are there?
d. Consider many examples of two numbers where the product of the two numbers is a multiple of 4. Look for patterns. How many odd numbers are there?
e. Consider many examples of two numbers where sum of two numbers is even. How many odd numbers are there?
f. Consider many examples of two numbers where sum of two numbers is odd. How many odd numbers are there?
g. Consider many examples of two numbers where ratio of two numbers is even. How many odd numbers are there?
h. Consider many examples of two numbers where ratio of two numbers is odd. How many odd numbers are there?
i. Suppose a standard 8x8 chessboard has two diagonally opposite corners removed, leaving 62 squares. Is it possible to place 31 dominoes of size 2x1 so as to cover all of these squares?

SOLUTIONS

a. There is one odd number.
b. There are either 0 or 2 odd numbers.
c. There is one odd numbers.
d. There are either 2 or 4 numbers.
e. There are either 0 or 2 odd numbers
f. There is 1 odd number.

g. There are either 0 or 1 odd numbers.
h. There are 0, 1 or 2 odd numbers.
i. This problem is hard if we think in terms of a chessboard without considering color of each square on the board to be different. However, if we consider the board to be consisting of alternatively black and white squares, we can argue that each domino will cover one black and one white squares. So, dominoes on the board will always cover exactly same number of black and white squares. However, if opposite corners are removed from the board, these corners are of the same color and hence the board has 32 squares of one color and 30 squares of another color. Hence, it is not possible to cover the board with dominoes.

Draw a smiley in the circle below to celebrate your progress on this exploration.



Exploration of Case-Based Reasoning

EXERCISES

1. I have four numbers: 2, 3, 4 and 6. In how many ways can I put these in a row of four squares?
2. Consider two 2 square cages in the first row of a 6x6 KenKen®. Each number from {2, 3, 4, 6} can occur at most once in these cages as 1 and 5 have already been assigned to the remaining squares in the row. One cage has a target of 12x. The other cage has a target of 1-. Which numbers are in the 1- cage?

SOLUTIONS

1. There are 4 x 3 x 2 x 1 = 24 ways to put in the numbers in four squares.
2. There are two possibilities (C1: 2, 6 C2: 3, 4) corresponding to a cage with a 12x target. There are two possibilities (B1: 2, 3 B2:3, 4) corresponding to the cage with a target of 1-. Considering all possible cases together, there is only one possibility that is consistent with the target numbers. This corresponds to C2 and B2: 2 and 6 in a 12x cage in addition to 3 and 4 in a 1- cage.

Cage 1	Cage 2	Use all numbers once
C1 3, 4	B1 2, 3	No
C1 3, 4	B2 3, 4	No
C2 2, 6	B1 2, 3	No
C2 2, 6	B2 3, 4	Yes

Draw a smiley in the circle below to celebrate your progress on this exploration.

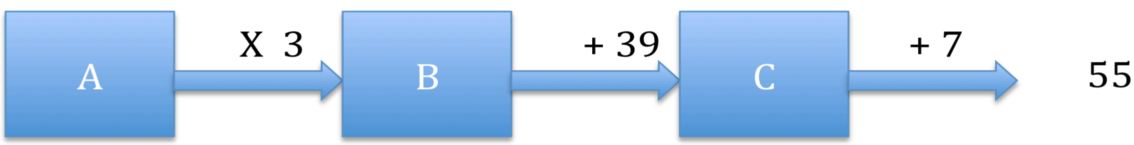
Exploration of Working Backward

Consider the following problem:

I have a list of ten numbers. In this list, the numbers 1 to 10 each occur exactly once, but not necessarily in increasing order. The sum of the first seven numbers is 39. The eighth number is 7. The ratio of the last two numbers is 2. What are the last two numbers?

Suppose the smaller of the numbers involved in the ratio is a. Now, numbers from 1 to 10 add up to 55. So, if I begin with a, add twice of the number to it, then add 39 to it and finally add 7 more to it, the result will be 55.

One strategy you can use here is to work backward. Let us describe what we described as a diagram:



To work backward, we try to determine C first.

As C + 7 = 55 and the inverse operation is C = 55 - 7 = 48.

Now, we will try to determine B.

As B + 39 = 48 and the inverse operation of is B = 48-39 = 9.

Finally, we will try to determine A.

As A multiplied by 3 is 9 and the inverse operation of multiplication is division, A is 9 divided by 3 = 3.

EXERCISES

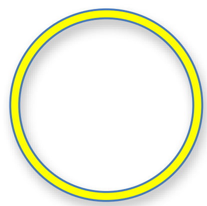
1. I start with a number. I multiply it by 4. I add 20 to it. I subtract 8 from it. I get 20. What was the number with which I started?
2. I start with a number. I multiply it by 4. I add 20 to it. I subtract 8 from it. I get 24. What was the number with which I started?
3. I start with a number. I multiply it by 4. I add 20 to it. I subtract 8 from it. I get 28. What was the number with which I started?

4. I start with a number. I add 10 to it. I multiply by 2. I add to 2 to it. The result is 26. What was the number with which I started?
5. I start with a number. I add 12 to it. I multiply by 2. I add 2 to it. The result is 32. What was the number with which I started?

SOLUTIONS

(1) 2 (2) 3 (3) 4 (4) 2 (5) 3

Draw a smiley in the circle below to celebrate your progress on this exploration.



Exploration of Arithmetic Sequence

In some of the previous explorations (sum difference), we saw that it is useful to know the sum of numbers from 1 to the maximum number allowed in the puzzle.

EXERCISES

(1) Look for patterns in the following table that lists a set of numbers in the first row and sum of these numbers in the second row.

1 to 2	1 to 3	1 to 4	1 to 5	1 to 6	1 to 7	1 to 8	1 to 9	1 to 10
3	6	10	15	21	28	36	45	55

SOLUTIONS

(1) A good strategy to look for patterns is to examine the differences.

Difference			1	1	1	1	1	1	1
Difference		3	4	5	6	7	8	9	10
	3	6	10	15	21	28	36	45	55
	1, 2	1–3	1–4	1–5	1–6	1–7	1–8	1–9	1–10

Another rule to remember is: When second level differences are constant, numbers are related to the square of n * n where n is the position of the number in the sequence. So, let's examine these by looking for patterns in the following table.

	2	3	4	5	6	7	8	9	10
n	2	3	4	5	6	7	8	9	10
n x n	4	9	16	25	36	49	64	81	100
Sum	3	6	10	15	21	28	36	45	55

After a little experimenting, we would find that the underlying pattern is the sum = (n x n + n) / 2.

Let us examine the sum.
$1 + 2 + 3 \ldots + 20$.
One way to obtain the answer to this sum is to use the formula mentioned above $(20 \times 20 + 20) / 2 = 210$.

Another method is to pair up the first number and the last number, the second number and the second to last number and so on. In each case, the sum turns out to be 21. Now, let us determine how many pairs of numbers there are. There are ten such pairs. Hence, the sum would be $21 \times 10 = 210$.

In an arithmetic sequence, the difference between one term and the next is a constant. For example, 1, 4, 7, 10, 13, 16, 19, 22, 25. In general, you could write an arithmetic sequence like this:

$\{a, a + d, a + 2d, a + 3d, \ldots\}$
where:
- a is the first term, and
- d is the difference between the terms (called the "common difference").

Consider the following problem about an arithmetic sequence: Find the sum of $1 + 3 + 5 + \ldots 49$.

Here, we can pair up the first number and the last number, the second number and the second to last number and so on. We have twelve such pairs all adding up to 50. The number in the middle is 25 by itself. So, the total would be $12 \times 50 + 25 = 625$.

A formula for arithmetic sequence is as follows:

$$a + (a + d) + (a + 2d) + (a + 3d) + \ldots(a + (n - 1) d) = n/2\, (a + a + (n - 1) d).$$

Draw a smiley in the circle below to celebrate your progress on this exploration.

KenKen Puzzles

3x3 Puzzles

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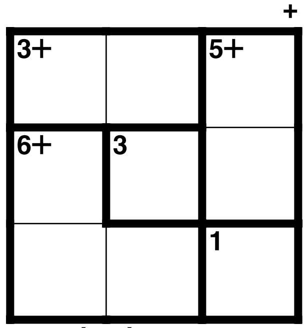

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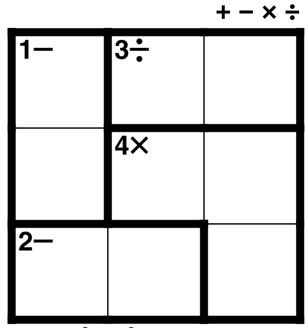

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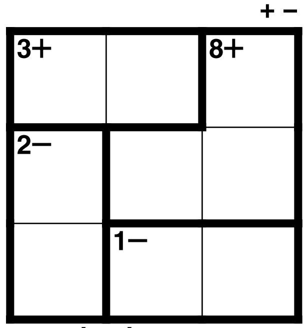

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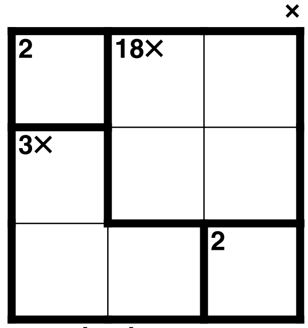

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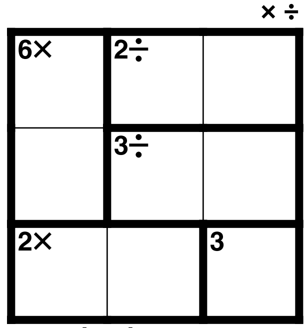

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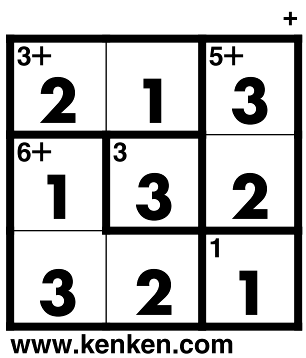
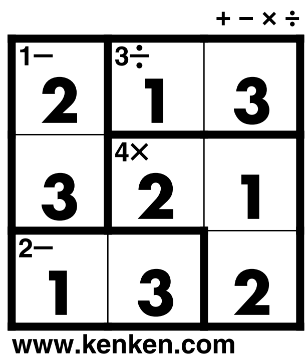
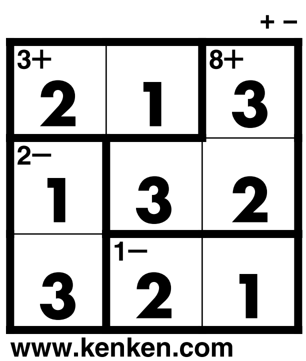
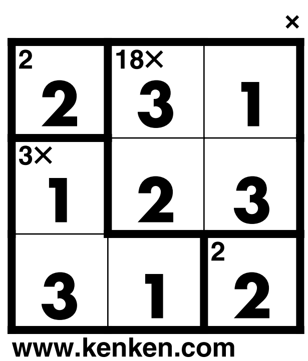
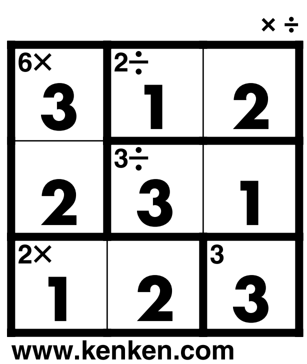
Solutions to 3x3 Puzzles





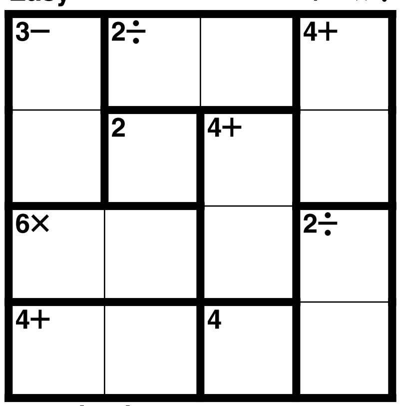






4x4 Puzzles

Easy + − × ÷



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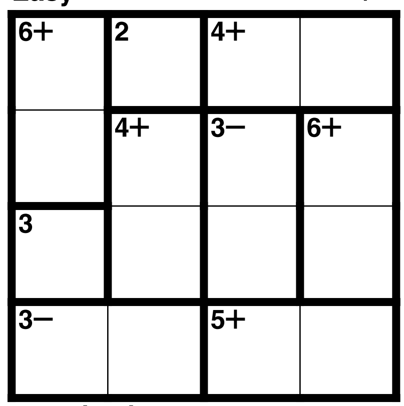
Easy + −



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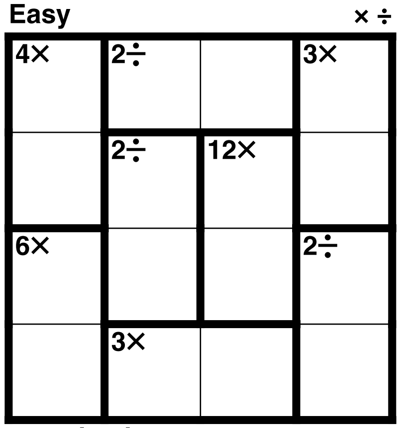
Easy × ÷



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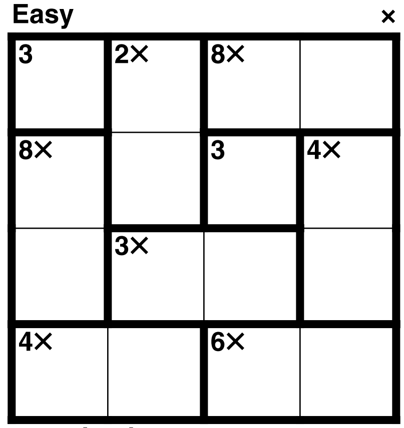
Easy ×



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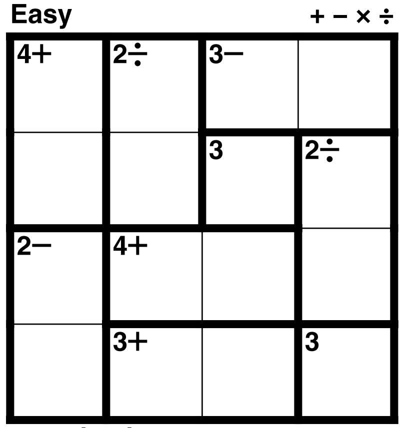
Easy + − × ÷



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Solution to 4x4 Puzzles

Easy + − × ÷

3− 1	2÷ 4	2	4+ 3
4	2 2	4+ 3	1
6× 2	3	1	2÷ 4
4+ 3	1	4 4	2

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Easy + −

6+ 4	2 2	4+ 3	1
2	4+ 3	3− 1	6+ 4
3 3	1	4	2
3− 1	4	5+ 2	3

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Easy × ÷

4× 1	2÷ 4	2	3× 3
4	2÷ 2	12× 3	1
6× 3	1	4	2÷ 2
2	3× 3	1	4

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Easy ×

3 3	2× 1	8× 4	2
8× 4	2	3 3	4× 1
2	3× 3	1	4
4× 1	4	6× 2	3

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Easy + − × ÷

4+ 3	2÷ 2	3− 4	1
1	4	3 3	2÷ 2
2− 2	4+ 3	1	4
4	3+ 1	2	3 3

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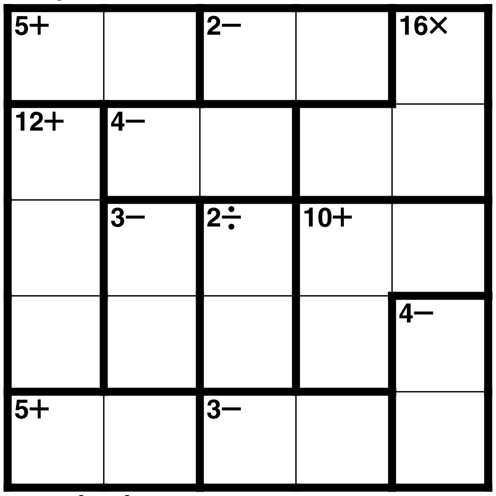
Easy + − × ÷



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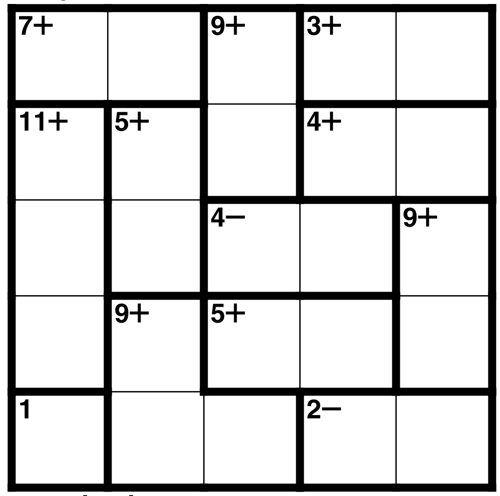
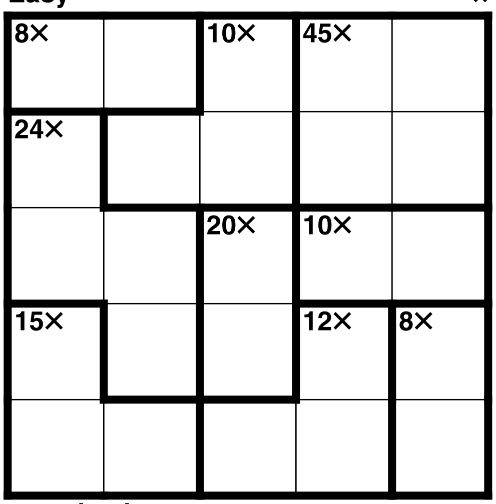
Easy + −



www.kenken.com

Easy ×



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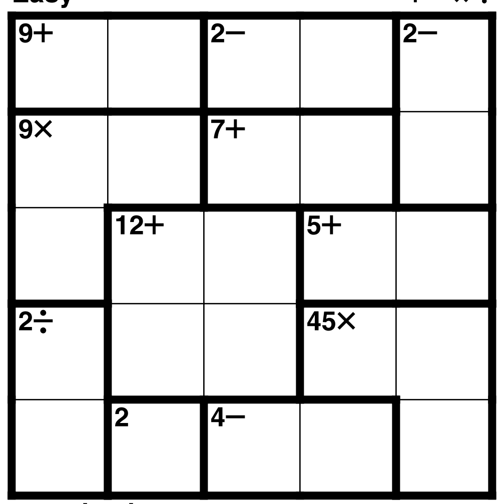
Easy + − × ÷



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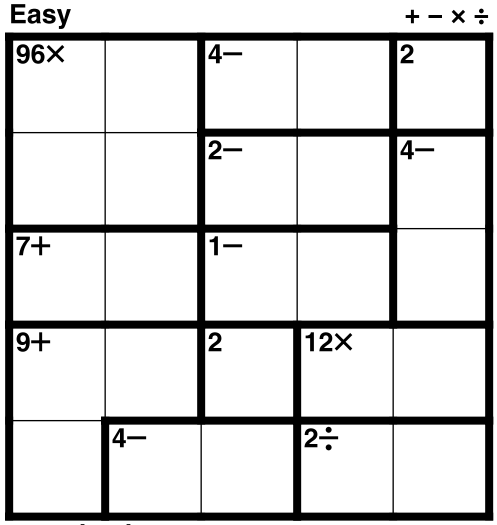
Easy + − × ÷



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Solution to 5x5 Puzzles

Easy + − × ÷

5+ 1	4	2− 3	5	16× 2
12+ 3	4− 1	5	2	4
5	3− 2	2÷ 1	10+ 4	3
4	5	2	3	4− 1
5+ 2	3	3− 4	1	5

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Easy + −

7+ 3	4	9+ 5	3+ 2	1
11+ 5	5+ 2	4	4+ 1	3
2	3	4− 1	5	9+ 4
4	9+ 1	5+ 2	3	5
1 1	5	3	2− 4	2

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Easy ×

8× 2	4	10× 1	45× 5	3
24× 4	5	2	3	1
1	3	20× 4	10× 2	5
15× 3	2	5	12× 1	8× 4
5	1	3	4	2

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Easy + − × ÷

9+ 5	4	2− 3	1	2− 2
9× 1	3	7+ 5	2	4
3	12+ 5	2	5+ 4	1
2÷ 2	1	4	45× 3	5
4	2 2	4− 1	5	3

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Easy + − × ÷

96× 4	3	4− 1	5	2 2
2	4	2− 3	1	4− 5
7+ 5	2	1− 4	3	1
9+ 1	5	2 2	12× 4	3
3	4− 1	5	2÷ 2	4

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Internet Resources for Math Problem Solving

In this chapter, we list some internet resources that can be helpful in learning creative problem solving skills. These websites are valid as of September 2012. A revised list of websites is available on matholympiad.info.

Websites for Math Puzzles

KenKen®: www.KenKen.com

Website for teachers to sign up for Free KENKEN® Classroom program: www.kenken.com/teachers

Jim Loy's Puzzles: http://www.jimloy.com/puzz/puzz.htm

Math is Fun Puzzles: http://www.mathsisfun.com/puzzles/index.html

Make 24:
http://www.sheppardsoftware.com/mathgames/make24_level_one/make24AS2_level1.htm

Make X puzzles:
http://www.sheppardsoftware.com/mathgames/makex/makeXAS2_level1.htm

Set game puzzle: http://www.setgame.com/set/puzzle_frame.htm

Sudoku: http://www.websudoku.com/

Numericana Recreational Math:
http://www.numericana.com/answer/recreational.htm

Websites for KenKen® Puzzle Techniques

Tom Davis, KenKen® for Teachers

http://www.geometer.org/mathcircles/KenKen.pdf

Prof. Reiter, KenKen® Strategies

http://math.uncc.edu/~hbreiter/JRMF/KenKenJR11.pdf

Websites for Math Competitions

MOEMS: www.moems.org

NOETIC Learning Math Contest: http://www.noetic-learning.com/mathcontest/details.jsp

Math Kangaroo: http://www.mathkangaroo.org/2010page/kangur/index.htm

World Math Day: http://www.worldmathday.com/

Online Math League: http://www.onlinemathleague.com/

Ole Miss Math Challenge: http://mathcontest.olemiss.edu/index.php

Math Bee: http://www.northsouth.org/public/main/home.aspx

Websites with Practice Problems

There are a number of websites with practice problems and worksheets. These include the following websites that were available as of August 2012:

Dr. Math Elementary Level Problems with Solutions

URL: http://mathforum.org/library/drmath/sets/elem_word_problems.html

Dr. Math Middle School Level Word Problems With Solutions

URL: http://mathforum.org/library/drmath/drmath.middle.html

Dr. Math Middle School Logic Problems with Solutions

URL: http://mathforum.org/library/drmath/sets/mid_logic.html

Dr. Math Middle School Puzzles with Solutions

URL: http://mathforum.org/library/drmath/sets/mid_puzzles.html

Math Forum PoW Sample Problems

URL: http://mathforum.org/pow/teacher/samples.html

Pacific Institute Sample Problems

URL: http://www.math.ubc.ca/~hoek/PIMS/Elmacon/Elmasam1.html

Math League Grade 4 Problems,

Math League Grade 4 Problem Solutions

http://www.mathleague.com/contests/Grade_4_2004-05_Contest.pdf

http://www.mathleague.com/contests/Grade_4_2004-05_Solutions.pdf

Math League Grade 5 Problems,

Math League Grade 5 Problem Solutions

http://www.mathleague.com/contests/Grade_4_2004-05_Contest.pdf

http://www.mathleague.com/contests/Grade_4_2004-05_Solutions.pdf

Math League Grade 6 Problems,

Math League Grade 6 Problem Solutions

http://www.mathleague.com/contests/Grade_5_2004-05_Contest.pdf

http://www.mathleague.com/contests/Grade_5_2004-05_Solutions.pdf

Math League Grade 7 Problems,

Math League Grade 7 Problem Solutions

http://www.mathleague.com/contests/Grade_6_2004-05_Contest.pdf

http://www.mathleague.com/contests/Grade_6_2004-05_Solutions.pdf

Math League Grade 8 Problems,

Math League Grade 8 Problem Solutions

http://www.mathleague.com/contests/Grade_7_2004-05_Contest.pdf

http://www.mathleague.com/contests/Grade_7_2004-05_Solutions.pdf

Mathcounts Drills

http://mathcounts.saab.org/mc.cgi

MOEMS Sample Contests

http://www.moems.org/sample.htm

Mathcounts 2011 Competitions

https://mathcounts.org/Page.aspx?pid=295

Math Kangaroo Practice Problems

Math Kangaroo Problem Answer Keys

http://www.mathkangaroo.org/2010page/Clark/clark/pdb/

http://www.mathkangaroo.com/2010page/kangur/AnswerKeys/AnswerKeys-links.htm