Goals for School Mathematics

The Report of the Cambridge Conference on School Mathematics
1963

Why?

The late 1950s and early 1960s witnessed an increasing amount of curriculum reform in mathematics. During this era there were few state frameworks and neither national standards nor national assessments to consider when developing mathematics curricula. Some projects focused on elementary and secondary mathematics while others focused on science and some integrated mathematics and science. While some duplication was happening, each of the projects was mapping out their own direction for change.

There were also pressures to teach more advanced topics and courses in the elementary and secondary schools (NCTM, 1970). Some university mathematicians were taking a keen interest in pre-college education. The reform efforts then underway, and associated newly developed curriculum materials, were a first step. However, many mathematicians and educators knew that the reform efforts and materials were not perfect, and not the end product of the much-needed changes. Inconsistencies in the reform efforts and the limiting of students' understanding of mathematics, sometimes as the result of teachers' limited content knowledge, were two of the shortcomings of the reform movements already present. In response to these emerging issues, a conference was organized "to deal with radical revisions in the mathematics curriculum" (NCTM, 1970, p. 291). The purpose was to (1) stimulate discussion of change in mathematics education beyond the limits currently placed on content and achievement, (2) make more radical revisions in the mathematics curriculum, and (3) look to the future to establish goals for mathematics education after the current reforms had run their course.

At a meeting in the summer of 1962, Professors Jerold Zacharias and William Ted Martin from the Massachusetts Institute of Technology, several Cambridge mathematicians, and representatives of the National Science Foundation (NSF), organized a conference to examine the changing mathematics curriculum at the elementary and secondary levels. Educational Services Incorporated was asked to help organize the conference, and financial support was obtained from NSF. On June 18, 1963, the conference, involving 29 outstanding mathematicians and natural scientists, convened in Cambridge, Massachusetts. The conference came to be known as the Cambridge Conference on School Mathematics, and its report, *Goals for School Mathematics,* was published in 1963.

Who?

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* These participants also belonged to the steering committee

What was produced?

A document proposing a rich mathematics curriculum was developed. It was intended to provide direction for mathematics curriculum developers and for the next generation of students to experience. Their work disregarded Piaget's view that certain concepts could not be learned until specific stages, and accepted Bruner's notion that any concept can be developed in some intellectually honest form for students at any age. In addition, the document ignored teacher preparation, and the focus was on creating an ambitious mathematics outline that was not restrained by educational psychology, pedagogy or real classrooms.

Section 1: Introduction

This section explained how the conference came to be, its purpose, how it was organized, and its charge.

Section 2: Broad Goals of the School Mathematics Curriculum

I. Acquisition of skills
- An increase in higher-level mathematical concepts means a decrease of something else—drill.
- Drill may be useful for students who are struggling with a particular skill, but this should be an exception to the rule.
- Practice of skills can be accomplished at the same time as new concepts are encountered.
- Another way to make room for new concepts is by reorganizing the subject matter.

II. Familiarity with mathematics
- From kindergarten on, geometry and arithmetic (then later geometry and algebra) should be developed simultaneously.
- Increased attention should be given to inequalities. (The classical curriculum almost exclusively dealt with equalities.)
- Familiarity with the ideas of calculus, algebra, and probability can be accomplished by establishing the global structure of mathematics.
- The elementary curriculum "should be understandable by virtually all students" (CCSM, 1963, p. 8).
- With the realization that fewer and fewer students elect to take mathematics in the later years, the most important concepts should be pushed down—especially probability and statistics.

III. Building self-confidence
- In contrast to authoritarian methods, a student can recreate formulas if they just learn some basic facts.
- Students should be given the explanations behind "short-cuts" so that they don't think they are just magical tricks.
- A major goal of mathematics education should be to build confidence in a student's own ability.

IV. Technical vocabulary and symbolism
- Terminology and notation are only as good as their effectiveness in communication; too much or too early of an introduction can obscure the concept.

V. Pure and applied mathematics
- Students should be made aware of when there are expectations for a logical, deductive argument, or when intuitive ideas are preferred.

- Both pure and applied mathematics should be approached intuitively as much as possible.
- "We hope that many problems can be found (we know a few) that read, 'Here is a situation—think about it—what can you say?'" (CCSM, 1963, p. 11).

VI. The power of mathematics

- The fallacy that mathematics is only concerned with "exact answers" needs to be exposed.
- High school students need to learn qualitative and probabilistic mathematical methods, which are becoming increasingly more important.

In this section, the committee discussed the level of mathematical attainment students should reach by the end of twelfth grade. To reach this end, drill was to be eliminated from all grades for the benefit of adding new material to the higher grade levels within which meaningful mathematics could be incorporated. Also discussed were beliefs as to the nature of the mathematics students should know as well as the role of mathematics in a liberal arts education, the character of language and notations, and both the power and the limitations of mathematics.

Section 3: Pedagogical Principles and Techniques

"In reaching the goals discussed above, the selection of germane material and the method of presentation are of prime importance. An omission of a subject from a student's curriculum can be made up readily later, in college or adult education courses, if the student has previously developed a sound approach to mathematics. On the other hand, improperly taught material may confuse the student's understanding of the facts, inhibit good mathematical reasoning, and lead to a dislike of the whole subject" (CCSM, 1963, p. 13).

I. Degrees of rigor

- Elementary students should experience an intuitive approach to the development of mathematical concepts before they are formally introduced.
- Spiraling brings the concept back again with increasing formality and depth, and emphasizes the unity and interdependence of mathematics.
- Rigor should never completely replace intuition to avoid killing creative potential.

II. The use of several approaches

- Fewer students will be left behind when several different approaches are used.

III. Development of skills

- Too much time has been spent on drill at the expense of new topics.

- It is more important that a student develop understanding of the justification of the manipulations, accuracy rather than speed, and be given time to check his or her answers.
- A student is not wrong for using a less efficient method (i.e. counting on fingers).

IV. Fostering independent and creative thinking
- The discovery approach is essential in developing children who are creative and independent thinkers—memorizing a mechanical procedure doesn't help the student.
- To aid discovery: communication, feedback, reinforcement, productive context, equipment, teacher's knowledge of when to interject, atmosphere absent of fear of wrong answers, and minimum authority of the teacher with respect to mathematical ideas.

V. Language and notation
- "A too early insistence on the use of the 'correct' word may well stifle the child's idea and will encourage parrot like responses" (CCSM, 1963, p. 20).
- Once the child has formulated an idea, the correct terminology and notation should be introduced to aid in the communication of ideas.

VI. The role of applications
- Internal and external applications of new concepts should be used to motivate the learning.

- Other pedagogical insights: Students should have opportunities to build Mathematical intuition before progressing to rigorous application of the previously learned material. To avoid stifling the creative potential of students, rigor should never completely replace intuition. Several approaches should be used to cover a given concept to insure that all students learn the material. Drill was again criticized, and standard algorithms were encouraged while still promoting alternate methods of computation. The idea of students discovering their own mathematics was enforced; however it should be balanced with direction from the teacher. Communication with peers about mathematics would foster greater thinking in the acceptance or rejection of classmates' conjectures in place of blind acceptance of the teacher's ideas. "Tricks" may be introduced, but only if their purpose and validity were fully explained. Interest in mathematics could be kindled through a study of history, the open nature of mathematics, games, and experiments. Mathematical language and notation should be continually introduced to aid student thinking and build connections as well as applications to provide motivation. Formal assessments should include problems that elicit higher order thinking and not merely measure skill level and knowledge of memorized facts.

Section 4: Some Overall Observations

I. The children
- Children will rise to the challenge—raise the bar and they will rise to meet it.
- If most children don't understand it, then we are presenting it wrong.

II. The teacher
- Proposed high school ". . . courses would include more content, on a higher level of sophistication, than most colleges now offer" (CCSM, 1963, p. 25).
- Since the teaching of ideas, not processes, was proposed, teachers needed to better understand the ideas themselves.

III. The problems
- Text writers should put more energy into writing the problems than writing the prose.
- Discovery problems should be encountered before the results are explicitly stated in the text.

IV. Testing, testers, and tests
- Using tests that existed at the time, it might have been hard to prove that this program worked.
- We should not teach to the test.
- Focus of assessments should be placed on student understanding instead of memorized facts.

This section focused on the committee's observations with regard to children, teachers, mathematical problems, and testing.

With regard to students, the committee members felt that mathematics which was taught "properly" could incorporate a higher level of mathematics and at a quicker pace than current instruction. A necessity may have existed to separate students by ability instead of the usual grade-levels as enrichment material may not have been sufficient for more advanced students.

The committee felt it had to keep its own ideas in check by monitoring student understanding. "We believe that no mathematical idea can be presented clearly unless it is presented correctly." (CCSM, 1963, p. 25) Therefore, mathematicians had to be involved at all levels of curriculum preparation to ensure proper presentation.

The curricula proposed in this document placed severe demands upon the teacher. Knowing that current teacher education programs were inadequate, the committee believed this could be remedied quickly and that future teachers should have greater content knowledge than what they would be teaching and be prepared to understand students' responses so as to not dismiss valid points that could have led to great mathematical discoveries.

The committee members commented on the difficulty of testing for deep levels of student knowledge over memorized facts and were concerned about the amount of time spent teaching to tests.

Section 5: Curriculum for Elementary School (K-6)

"The objective for mathematics instruction in the elementary grades is familiarity with the real number system and the main ideas of geometry" (CCSM, 1963, p. 31). The content taught at these levels was considered "pre-mathematics."

This section was divided into two subsections: K-2 and 3-6 expectations, each outlined the committee's hope for the evolution of mathematics instruction.

The Earliest Grades, K-2

This subsection was divided into 5 parts: the real number system, geometry, logic and set theory/function, applications, and general remarks.

A subset of the recommended items includes the following examples:

1) *The idea of inequalities and the symbols, < or >.*
2) *The number line, including negatives from the beginning.*
3) *The use of "crossed" number lines to form Cartesian Coordinates.*
4) *Use of a straightedge and compass to do the standard geometric constructions such as comparing segments or angles, bisecting a segment or angle, etc.*

(CCSM, 1963, pp. 33-34)

The Committee recommended that the focus of the study of real numbers be on developing insight instead of learning algorithms, and on understanding over calculating. Use of "physical equipment" was also strongly encouraged

Grades 3 through 6

This subsection was divided into six parts: the real number system, geometry, logic and foundations, theory of real functions, applications, and longer projects for students.

A subset of the recommended items includes the following examples:

1) *Arithmetic of signed numbers.*
2) *Study of 2x2 matrices; isomorphisms of a subset 2x2 matrices with real numbers; identities for matrices; simple matrix inversions.*
3) *Integral exponents, both positive and negative.*
4) *Conic sections.*
5) *Vectors, possibly including some statics or linear kinematics.*

(CCSM, 1963, pp. 36-38)

Many of the topics were recommended by the committee as a way to prepare students for the mathematics they would face at the secondary level. However, the level of complexity involved with each topic was left up to the teacher. Longer projects were suggested as a way to introduce students to more involved, multi-tiered problems students were likely to experience in their futures.

Section 6: Curriculum for Grades 7-12

Members of the Committee could not always reach a consensus and therefore, two secondary curricula were proposed.

The first proposal:

Grades 7/8:	Part 1 – Algebra
	Part 2 – Probability
Grade 9:	Geometry
Grade 10:	Part 1 – Geometry, Topology, and Algebra
	Part 2 – Linear Algebra
Grades 11/12:	Analysis

The second proposal:

Grades 7/8:	Part 1 – Algebra and Geometry
	Part 2 – Probability
Grade 9:	Part 1 – Introductory Calculus
	Part 2 – Algebra and Geometry
Grade 10:	Analysis, Probability, and Algebra
Grades 11/12:	Analysis

How the proposals treated certain concepts:

Algebra

There was an agreement by all that review of properties of real numbers was necessary at the beginning of seventh grade.

Proposal 1: polynomial forms over fields, applications of polynomial functions, difference operators, complex functions, tangent of the graph of a polynomial.

Proposal 2: emphasis on polynomial functions, trigonometric, functions studied earlier, different presentation of complex numbers.

Calculus

Proposal 1: waited until Grades 11 and 12 to give the students a thorough presentation of calculus.

Proposal 2: introduced some elements of calculus in grade 9 that were felt to be of importance for all students, not just those completing all twelve years of mathematics.

Here is a subset of the recommended items:

<u>Proposal 1</u>

Grades 7/8

Part 1

1) *Review of properties of numbers.*
2) *Ring of polynomials over a field.*
3) *Complex numbers over a residue class of polynomials mod* x^2+1
4) *Derivative of a polynomial.*

Part 2

1) *Conditional probability, independence.*
2) *Random variables and their distribution.*
3) *Poisson distribution.*

(CCSM, 1963, p. 43)

Grade 9

1) *Intuitive and synthetic geometry to the Pythagorean Theorem.*
2) *Complex numbers and rotations in the plane, trigonometry.*
3) *Vector space of n dimensions.*

(CCSM, 1963, p. 44)

Grade 10

Part 1

1) *Geometry and Topology of the Complex plane.*
2) *Geometry of complex numbers, linear fractional transformations, mappings by elementary functions, stereographic projection.*
3) *Fundamental theorem of algebra, winding number, location of roots.*

Part 2

1) *Simultaneous linear equations, linear mappings, matrices.*
2) *Equivalence of matrices, change of bases, and matrices of a transformation.*
3) *Inner products and orthogonal transformations.*

(CCSM, 1963, p. 44)

Grades 11/12

1) *Limits of functions, continuous functions.*
2) *Linear differential equations with constant coefficients.*

3) *Taylor series, indeterminate forms.*
4) *Probability for functions of several variables.*

(CCSM, 1963, pp. 44-45)

Proposal 2

Grades 7/8

Part 1

1) *Review of properties of numbers.*
2) *Logic of formal proofs discussed, axiomatic development of Euclidean geometry of two and three dimensions.*
3) *Vectors in two and three dimensions.*
4) *Complex numbers, possible introduction to logarithms.*

Part 2

Binomial Theorem, combinatorial problems.

(The rest of the proposed ideas were the same as Proposal 1.)

(CCSM, 1963, p. 45)

Grade 9

Part 1

1) *Limits of functions and continuity (lightly).*
2) *Derivatives of polynomials, sines and cosines, sums and products.*
3) *Antiderivatives, definite integral, and area.*
4) *The Mean Value Theorem, Fundamental Theorem of Calculus, applications.*

Part 2

1) *Linear equations and planes.*
2) *Rigid motions of space, linear and affine transformations and matrices, determinants, solutions of linear systems.*
3) *Quadratic forms, diagonalization, conics, and quadrics.*

(CCSM, 1963, pp. 45-46)

Grade 10*

1) *Infinite sequences and series of real and complex numbers, absolute and unconditional convergence, power series*
2) *Probability for countable sample spaces.*
3) *Linear algebra, subspaces, bases, dimension, coordinates, linear transformations and matrices, systems of equations, determinants, quadratic forms, diagonalization.*

*These comprised the entirety of the 10th-grade recommendations.

(CCSM, 1963, p. 46)

Grades 11/12

1) *Limits of function, continuity.*

2) *Mean Value Theorem and its consequences.*
3) *Differential equations.*
4) *Differential geometry of curves in space.*

(CCSM, 1963, p.46)

Appendices

The document included a number of appendices where specific examples of ideas and activities mentioned in the proposals were illustrated in some depth. For example, Appendix B was devoted to showing how logarithms might be constructed in a meaningful way in the elementary school.

Significance and Criticism of the Report

Some mathematicians felt the ideas set forth in the Report were a good place for future research to begin, while others criticized its elitist feel with bias toward the more capable student. Either way it provoked lively discussion.

Example: In a review printed in the *Mathematics Teacher*, Marshall Stone, a pure mathematician from the University of Chicago, was curious about the lack of mention by the participants of a similar program already in place in Europe and also highly criticized the structure of the elementary school program. He observed that none of the participants were familiar with elementary studies at the Conference and doubted that discovery learning and the spiral nature of the curriculum would solve the problems of those struggling with mathematics. Stone also did not like the noted absence of references to other committees and curriculum programs such as UICSM and SMSG. Regarding the structure of the secondary program, Stone scrutinized the content as a shift from college mathematics to high school without any modification, and felt that more time should have been spent structuring the program to suit student needs. Lastly, Stone strongly stated his concern over the insufficiencies of teacher preparation programs, which were not dealt with at the conference.

The Steering Committee met again in 1964 and discussed contacting other organizations involved in curriculum reform and holding conferences to deal with other topical issues. One such conference was on teacher training and produced *Goals for Mathematical Education of Elementary School Teachers*, which tackled the issue of producing capable teachers who could teach the new subject matter being created during the new math era.

Irving Adler, a mathematics educator wrote an article published in the *Mathematics Teacher* where he stated that the content set forth in the conference report was not meant as a blueprint for curriculum but as a guide for others from which a blueprint could be created. Unlike Stone, Adler praised the introduction of abstract concepts in the elementary level and believed that their presence would simplify learning. He too, like Stone, noted that the inadequacies of teacher preparation programs would make implementation of the ideas contained in the report impossible. However, he felt that the suggestions were not fantastical, and made suggestions that would assist in the realization of the report's content including dissecting the report, examining the current curricular movement overseas, furthering the study of mathematics unknown or unfamiliar to educators, and fostering discussion of the report amongst current teachers.

The report had some broad impact on subsequent mathematics curriculum and curriculum-related concerns. Due to the increased expectations of students, concern for lower ability students was raised. The recommendations accentuated the problem of *What mathematics is appropriate for whom?* The report also increased awareness of the need for better teacher preparation. Additionally, it inspired a closer look at teaching methodology such as the discovery method. "Newer concepts and processes, such as mathematical-model building, are acquiring significance for the schools" (NCTM, 1970, p. 297). Downward movement of course content was stimulated, and some courses were developed that were more integrated and less compartmentalized.

The Cambridge Conference stimulated and influenced several curriculum projects, including the Comprehensive School Mathematics Project (directed by Burt Kaufman) and Secondary School Mathematics Curriculum Improvement Study (directed by Howard Eves).

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